

Modulo Orientations with Bounded Out-Degrees and Modulo Factors with Bounded Degrees

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Abstract

Let G be a graph, let k be a positive integer, and let $p : V(G) \rightarrow Z_k$ be a mapping with $|E(G)| \stackrel{k}{\equiv} \sum_{v \in V(G)} p(v)$. In this paper, we show that if G is $(3k-3)$ -edge-connected, then G has an orientation such that for each vertex v , $d_G^+(v) \stackrel{k}{\equiv} p(v)$ and

$$\lfloor \frac{d_G(v)}{2} \rfloor - (k-1) \leq d_G^+(v) \leq \lceil \frac{d_G(v)}{2} \rceil + (k-1).$$

Also, we show that if G contains $(2k-2)$ edge-disjoint spanning trees, then G has an orientation such that for each vertex v , $d_G^+(v) \stackrel{k}{\equiv} p(v)$ and

$$k/2 - 1 \leq d_G^+(v) \leq d_G(v) - k/2 + 1.$$

This result reduces the required edge-connectivity of several results toward decomposing a graph into isomorphic copies of a fixed tree. Next, we conclude that if G is a $(3k-3)$ -edge-connected bipartite graph with the bipartition (A, B) , then it has a factor H such that for each vertex v , $d_H(v) \stackrel{k}{\equiv} f(v)$ and

$$\lfloor \frac{d_G(v)}{2} \rfloor - (k-1) \leq d_H(v) \leq \lceil \frac{d_G(v)}{2} \rceil + (k-1),$$

where $f : V(G) \rightarrow Z_k$ is a mapping with $\sum_{v \in A} f(v) \stackrel{k}{\equiv} \sum_{v \in B} f(v)$. Finally, we investigate decomposition of highly edge-connected graphs into factors with bounded degrees with many edge-disjoint spanning trees and deduce that every 4-edge-connected graph G has a spanning Eulerian subgraph whose degrees are close to $d_G(v)/2$. As a consequence, every 4-edge-connected 10-regular graph has a spanning Eulerian subgraph whose degrees lie in the set $\{4, 6\}$.

Keywords: Modulo orientation; out-degree; modulo factor; vertex degree; spanning tree; spanning Eulerian; connected factor; out-branching.

1 Introduction

In this article, all graphs have no loop, but multiple edges are allowed. Let G be a graph. The vertex set, the edge set, the maximum degree, and the minimum degree of vertices of G are denoted by $V(G)$, $E(G)$, $\Delta(G)$, and $\delta(G)$, respectively. Throughout this article, we denote by $d_G(v)$ the degree of a vertex v in the graph G , whether G is directed or not. If G has an orientation, the out-degree and in-degree of v are denoted by $d_G^+(v)$ and $d_G^-(v)$. For a vertex set A of G , the number of edges of G with exactly one end in A is denoted by $d_G(A)$. Also, we denote by $e_G(A)$ the number of edges with both ends in A and denote by $e_G(A, B)$ the number of edges with one end in A and one end in B , where B is a vertex set. For notational simplicity, we write A^c for the vertex set $V(G) \setminus A$. Let k be a positive integer. The cyclic group of order k is denoted by Z_k . An orientation of G is said to be p -orientation, if for each vertex v , $d_G^+(v) \stackrel{k}{\equiv} p(v)$, where $p : V(G) \rightarrow Z_k$ is a mapping with $|E(G)| \stackrel{k}{\equiv} \sum_{v \in V(G)} p(v)$. Note that for two rational numbers a and b , we say that $a \stackrel{k}{\equiv} b$, if $a - b$ is an integer and is divisible by k . Define λ_k to be a positive integer such that every λ_k -edge-connected graph G admits a p -orientation where $p : V(G) \rightarrow Z_k$ is a mapping with $|E(G)| \stackrel{k}{\equiv} \sum_{v \in V(G)} p(v)$. Throughout this article, let m denote a nonnegative integer. A graph G is called m -tree-connected, if it contains m edge-disjoint spanning trees. Note that by the result of Nash-Williams [25] and Tutte [35] every $2m$ -edge-connected graph is m -tree-connected. A graph is termed essentially λ -edge-connected, if all edges of any edge cut of size strictly less than λ are incident to a vertex. An f -factor, refers to a spanning subgraph H such that for each vertex v , $d_H(v) \stackrel{k}{\equiv} f(v)$, where $f : V(G) \rightarrow Z_k$. For a connected bipartite graph G , we say that a mapping $f : V(G) \rightarrow Z_k$ is compatible by G , if $\sum_{v \in A} f(v) \stackrel{k}{\equiv} \sum_{v \in B} f(v)$, where (A, B) is the unique bipartition of G . The bipartite index $bi(G)$ of a graph G is the smallest number of all $|E(G) \setminus E(H)|$ taken over all bipartite spanning subgraphs H . For a graph G with the vertex u , we denote by χ_u the mapping $\chi_u : V(G) \rightarrow \{0, 1\}$ such that $\chi(u) = 1$ and $\chi(v) = 0$ for all vertices v with $v \neq u$. Two different edges are called parallel, if have the same end vertices. For two edges xu and uy incident with the vertex u , lifting of xu and uy is an operation that removes xu and uy , and also add a new edge xy when xu and yu are not parallel. Also, if one of xu and yu , as xu , is directed, then we direct xy toward y when xu is toward u , and direct xy away from y when xu is away from u . Let G' be a graph obtained from G by lifting two edges xu and uy . Conversely, the reverse of lifting operation on xu and uy of the graph G' , derives the original graph G from G' , if xu and uy are not parallel; otherwise derives G' . Every orientation of G' induces an orientation for G by orienting xu and uy in a opposite direction from u , and if xu and yu are not parallel and xy is directed from x to y in G' , then in G the edge xu is directed from x to u and the edge uy is directed from u to y . Note that if a graph L obtained from G by alternatively lifting operations, then a given arbitrary orientation of L induces an orientation for G such for each vertex v , $d_G^+(v) = d_L^+(v) + (d_G(v) - d_L(v))/2$. Note that every edge xy of L is corresponded to the unique trail P_{xy} in G with the end vertices x and y such that the edge xy was obtained from P_{xy} by alternatively lifting operations and P_{xy} can be obtained from xy by reversing these lifting operations, moreover such trails are edge-disjoint.

In 2012 Thomassen constructed the following theorem on modulo orientations.

Theorem 1.([30]) *Let G be a graph, let k be an integer, $k \geq 3$, and let $p : V(G) \rightarrow Z_k$ be a mapping with $|E(G)| \equiv \sum_{v \in V(G)} p(v) \pmod k$. If G is $(2k^2 + k)$ -edge-connected, then G has a p -orientation.*

Later, Lovász, Thomassen, Wu, and Zhang (2013) refined Theorem 1 by reducing the quadratic bound $(2k^2 + k)$ down to a linear bound as the following theorem. In this paper, we refine their result by pushing the required edge-connectivity down to $3k - 3$, even for even numbers k , and strengthen it by giving a sharp bound on out-degrees. In particular, we strengthen the recent result in [15] toward this concept which improves the required edge-connectivity of several results in [2, 5, 6, 23, 28, 29, 31] toward decomposing a graph into isomorphic copies of a fixed tree.

Theorem 2.([20]) *Let G be a graph, let k be an integer, $k \geq 3$, and let $p : V(G) \rightarrow Z_k$ be a mapping with $|E(G)| \equiv \sum_{v \in V(G)} p(v) \pmod k$. If G is λ -edge-connected, then G has a p -orientation, where $\lambda = 3k - 3$ when k is odd and $\lambda = 3k - 2$ when k is even.*

In 2014 Thomassen introduced the concept of modulo factors and formulated the following theorem. Recently, Bensmail, Merker, and Thomassen [4] applied it with a weaker version based on Theorem 2 to deduce that every 16-edge-connected bipartite graph admits a decomposition into at most two locally irregular subgraphs. Fortunately, by the above-mentioned improvement for Theorem 2, the following theorem is also holds even for even numbers k and can refine their result.

Theorem 3.([33]) *Let G be a bipartite graph, let k be an integer, $k \geq 3$, and let $f : V(G) \rightarrow Z_k$ be a compatible mapping. If G is $(3k - 3)$ -edge-connected, then G has an f -factor.*

In Section 3, we generalize Theorem 3 for investigating m -tree-connected f -factors whose degrees are around $d_G(v)/2$ imposed by some bounds depending on k and m . For the special case $k = 2$, we refine the following theorem due to Jaeger (1979) and Catlin (1988) by concluding that every 4-edge-connected graph G has a spanning Eulerian subgraph whose degrees are close to $d_G(v)/2$.

Theorem 4.([9, 16]) *Every 4-edge-connected graph has a spanning Eulerian subgraph.*

Finally, we generalize the recent results in [1] to prove the assertions in Section 3. Indeed, we present similar but slightly more complicated versions of the following theorem. The proofs are postponed until Section 4.

Theorem 5. *Every $(2m_1 + 2m_2)$ -edge-connected graph G with $m_1 + m_2 \geq 1$ can be decomposed into*

two factors G_1 and G_2 such that each G_i is m_i -tree-connected and for each vertex v ,

$$\begin{aligned} \lfloor \frac{d_G(v)}{2} \rfloor - m_2 &\leq d_{G_1}(v) \leq \lceil \frac{d_G(v)}{2} \rceil + m_1, \\ \lfloor \frac{d_G(v)}{2} \rfloor - m_1 &\leq d_{G_2}(v) \leq \lceil \frac{d_G(v)}{2} \rceil + m_2. \end{aligned}$$

2 Modulo orientations with bounded out-degrees

In this section, we investigate orientations modulo k whose out-degrees are restricted to predetermined best possible intervals. We begin with orientations modulo 2. Later, we study a conjecture on orientations modulo k on graphs with edge-connectivity at least $2k - 1$, and also provide a solution for it in graphs with edge-connectivity at least $3k - 3$. Finally, we push the required edge-connectivity down to $2k - 2$ in graphs with many edge-disjoint spanning trees in compensation for greater out-degree bounds.

2.1 Preliminaries

In this subsection, we state two well-known propositions depending on lifting operations which can easily be proved by a combination of Mader's Theorem [13, 22] and Menger's Theorem, see [23, 26].

Proposition 1 *Let G be a λ -edge-connected graph with $\lambda \geq 2$. If $u \in V(G)$ and $d_G(u) \geq \lambda + 2$, then there are two edges incident with u such that by lifting them the resulting graph is still λ -edge-connected.*

Proposition 2 *Let G be a λ -edge-connected graph with $\lambda \geq 2$ and $|V(G)| \geq 2$. If $u \in V(G)$ and $d_G(u)$ is even, then we can alternatively lift $d_G(u)/2$ disjoint pair of edges incident with u such that the resulting graph H with $V(H) = V(G) \setminus \{u\}$ is still λ -edge-connected.*

2.2 Orientations modulo 2

In 2012 Thomassen observed that edge-connectedness 1 is sufficient for a graph to have a p -orientation modulo 2, however this edge-connectedness cannot guarantee that out-degrees are strictly more than zero even in graphs with large degrees.

Lemma 1.([30]) *Let G be a graph and let $p : V(G) \rightarrow \mathbb{Z}_2$ be a mapping with $|E(G)| \stackrel{2}{\equiv} \sum_{v \in V(G)} p(v)$. If G is connected, then G has a p -orientation.*

Here, we show that edge-connectedness 2 is sufficient for a graph to have a p -orientation modulo 2, where out-degrees fall in predetermined short intervals.

Theorem 6. *Let G be a graph and let $p : V(G) \rightarrow \mathbb{Z}_2$ be a mapping with $|E(G)| \stackrel{2}{\equiv} \sum_{v \in V(G)} p(v)$. If G is 2-edge-connected, then G has a p -orientation such that for each vertex v ,*

$$\lfloor \frac{d_G(v)}{2} \rfloor - 1 \leq d_G^+(v) \leq \lceil \frac{d_G(v)}{2} \rceil + 1.$$

Furthermore, for an arbitrary vertex z_0 , $d_G^+(z_0)$ can be assigned to any plausible integer value in whose interval.

Proof. By induction on the sum of all $d_G(v) - 3$ taken over all vertices v with $d_G(v) \geq 4$. First, assume that for each vertex v , $d_G(v) \leq 3$. For $|V(G)| = 1$, there is nothing to prove. So, suppose $|V(G)| \geq 2$. It is not hard to check that there is an edge set E incident with z_0 such that $G - E$ is connected, where $|E| = 1$ when z_0 has even degree and $|E| = 2$ when z_0 has odd degree; see Lemma 9. Orient the edge(s) of E away from z_0 , if the goal on z_0 is that $d_G^+(z_0) \geq \lceil \frac{d_G(z_0)}{2} \rceil$, and toward z_0 , if the goal on z_0 is that $d_G^+(z_0) \leq \lfloor \frac{d_G(z_0)}{2} \rfloor$. By applying Lemma 1 to the graph $G - E$, the pre-orientation of E can be extended to a p -orientation of G satisfying the theorem. Now, assume that for a vertex u , $d_G(u) \geq 4$. By Proposition 1, there are two edges xu and yu of G incident with u such that by lifting them the resulting graph H is still 2-edge-connected. If xu and yu are parallel, define $p' = p - \chi_u - \chi_x$; otherwise, define $p' = p - \chi_u$. Now, by the induction hypothesis, H has a p' -orientation such that for each vertex v , $\lfloor \frac{d_H(v)}{2} \rfloor - 1 \leq d_H^+(v) \leq \lceil \frac{d_H(v)}{2} \rceil + 1$, because of $|E(H)| \stackrel{2}{\equiv} \sum_{v \in V(H)} p'(v)$. This orientation of H induces an orientation for G such that $d_G^+(u) = d_H^+(u) + 1$, $d_G^+(v) = d_H^+(v)$ for any $v \in V(G) \setminus \{u, x\}$, and

$$d_G^+(x) = \begin{cases} d_H^+(x) + 1, & \text{if } xu \text{ and } yu \text{ are parallel;} \\ d_H^+(x), & \text{if } xu \text{ and } yu \text{ are not parallel.} \end{cases}$$

For instance, for the vertex u , we have

$$\lfloor \frac{d_G(u)}{2} \rfloor - 1 = (\lfloor \frac{d_H(u)}{2} \rfloor - 1) + 1 \leq d_G^+(u) \leq (\lceil \frac{d_H(u)}{2} \rceil + 1) + 1 = \lceil \frac{d_G(u)}{2} \rceil + 1.$$

The extra condition on $d_G^+(z_0)$ can be obtained by giving an appropriate condition on $d_H^+(z_0)$. It is easy to see that the orientation of G is a p -orientation satisfying the desired properties. Hence the theorem holds. \square

2.3 Graphs with edge-connectivity at least $2k - 1$

Note that if tree-connectedness k would be sufficient for a graph G to have a p -orientation modulo k (see Conjecture 2 in [15]), then this tree-connectedness cannot guarantee that out-degrees are strictly less than $(1 - \frac{1}{k})d_G(v)$ even in graphs with large degrees. It looks natural that Theorem 6 could be generalized by replacing edge-connectedness $2k$. We feel that this edge-connectedness can also be improved as the following qualitative conjecture.

Conjecture 1. Let G be a graph, let k be an integer, $k \geq 3$, and let $p : V(G) \rightarrow Z_k$ be a mapping with $|E(G)| \stackrel{k}{\equiv} \sum_{v \in V(G)} p(v)$. If G is $(2k-1)$ -edge-connected, then G has a p -orientation such that for each vertex v ,

$$\lfloor \frac{d_G(v)}{2} \rfloor - (k-1) \leq d_G^+(v) \leq \lceil \frac{d_G(v)}{2} \rceil + (k-1).$$

Furthermore, for an arbitrary vertex z_0 , $d_G^+(z_0)$ can be assigned to any plausible integer value in whose interval.

In 1992 Jaeger, Linial, Payan, and Tarsi [18] conjectured that the assignment of $\lambda_k = 2k-1$ is admissible, for $k=3$, in terms of group-connectivity of graphs. Later, Lai [19] extend it for odd numbers k . Surprisingly, these conjectures can mainly imply the above-mentioned conjecture with respect to the following theorem.

Theorem 7. Let k be an integer with $k \geq 3$. If the assignment of $\lambda_k = 2k-1$ is admissible, then Conjecture 1 is true (without considering the extra condition on z_0).

Proof. By induction on the sum of all $d_G(v) - 2k + 1$ taken over all vertices v such that either $d_G(v) \geq 2k+1$ or $d_G(v) = 2k$ and $p(v) \stackrel{k}{\equiv} 0$. For $|V(G)| \leq 2$, the proof is straightforward. So, suppose $|V(G)| \geq 3$. If for each vertex v , either $d_G(v) = 2k-1$ or $d_G(v) = 2k$ and $p(v) \not\stackrel{k}{\equiv} 0$, then the proof can be obtained directly from the assumption. Suppose first that there is a vertex u with $d_G(u) \geq 2k+1$. By Proposition 1, there are two edges xu and yu of G incident with u such that by lifting them the resulting graph H is still $(2k-1)$ -edge-connected. If xu and yu are parallel, define $p' = p - \chi_u - \chi_x$; otherwise, define $p' = p - \chi_u$. By the induction hypothesis, H has a p' -orientation such that for each vertex v , $\lfloor \frac{d_H(v)}{2} \rfloor - (k-1) \leq d_H^+(v) \leq \lceil \frac{d_H(v)}{2} \rceil + (k-1)$, because of $|E(H)| \stackrel{k}{\equiv} \sum_{v \in V(H)} p'(v)$. This orientation of H induces an orientation for G such that $d_G^+(u) = d_H^+(u) + 1$, $d_G^+(v) = d_H^+(v)$ for any $v \in V(G) \setminus \{u, x\}$, and

$$d_G^+(x) = \begin{cases} d_H^+(x) + 1, & \text{if } xu \text{ and } yu \text{ are parallel;} \\ d_H^+(x), & \text{if } xu \text{ and } yu \text{ are not parallel.} \end{cases}$$

For instance, for the vertex u , we have

$$\lfloor \frac{d_G(u)}{2} \rfloor - (k-1) = \lfloor \frac{d_H(u)}{2} \rfloor - (k-1) + 1 \leq d_G^+(u) \leq \lceil \frac{d_H(u)}{2} \rceil + (k-1) + 1 = \lceil \frac{d_G(u)}{2} \rceil + (k-1).$$

It is easy to see that the orientation of G is a p -orientation satisfying the theorem. Now, suppose that $\Delta(G) \leq 2k$ and there is a vertex u with $d_G(u) = 2k$ and $p(u) \stackrel{k}{\equiv} 0$. By Proposition 2, we can alternatively lift the edges incident with u such that the resulting graph H with $V(H) = V(G) \setminus \{u\}$ is still $(2k-1)$ -edge-connected. Since $\Delta(G) \leq 2k$ and H is $(2k-1)$ -edge-connected, any pair of edges which are lifted are not parallel. By the induction hypothesis, H has a p -orientation such that for each vertex v with $v \neq u$, $\lfloor \frac{d_H(v)}{2} \rfloor - (k-1) \leq d_H^+(v) \leq \lceil \frac{d_H(v)}{2} \rceil + (k-1)$, because of $|E(H)| \stackrel{k}{\equiv} \sum_{v \in V(H)} p(v)$.

This orientation of H induces an orientation for G such that $d_G^+(u) = k$ and $d_G^+(v) = d_H^+(v)$ for each vertex v with $v \neq u$. For the vertex u , we have

$$0 < \lfloor \frac{d_G(u)}{2} \rfloor - (k-1) \leq d_G^+(u) \leq \lceil \frac{d_G(u)}{2} \rceil + (k-1) < 2k.$$

It is easy to see that the orientation of G is a p -orientation satisfying the desired properties. Hence the theorem holds. \square

Remark 1. By applying the same arguments in the proof of the above-mentioned theorem, for an arbitrary λ_k we could only show that every λ_k -edge-connected G has a p -orientation such that for each vertex v , $\lfloor \frac{d_G(v) - \lambda_k}{2} \rfloor \leq d_G^+(v) \leq \lceil \frac{d_G(v) + \lambda_k}{2} \rceil$, where $p : V(G) \rightarrow Z_k$ is a mapping with $|E(G)| \stackrel{k}{\equiv} \sum_{v \in V(G)} p(v)$.

2.4 Graphs with edge-connectivity at least $3k - 3$

In this subsection, we provide a solution for Conjecture 1 in graphs with edge-connectivity at least $3k - 3$. We follow with the same innovative ideas that appeared in [20] and retain the same arguments, while modifications are inserted. The proof is based on defining a set function α whose values lie in the set $\{0, \pm 1/2, \dots, \pm k/2\}$. It is inspired by the set function $\tau(A)$ in [20] and the set function $t(A)$ in [30]. More precisely, for odd integers k , $2\alpha(A) = \tau(A)$, and for odd and even integers k , $2|\alpha(A)| = t(A)$.

Let G be a graph, let k be an integer, $k \geq 3$, and let $p : V(G) \rightarrow Z_k$ be a mapping with $|E(G)| \stackrel{k}{\equiv} \sum_{v \in V(G)} p(v)$. For each vertex v , take $\alpha(v)$ to be a rational number such that $\alpha(v) \in \{0, \pm 1/2, \dots, \pm k/2\}$ and $\alpha(v) \stackrel{k}{\equiv} p(v) - d_G(v)/2$. In intuitive terms, $|\alpha(v)|$ specifies the distance between two points $p(v)$ and $d_G(v)/2$ on a circle whose circumference is k , and the sign of $\alpha(v)$ determines the position of $p(v)$ with respect to $d_G(v)/2$. Thus, it is intuitively clear and not difficult to show that $\alpha(v)$ is unique unless $\alpha(v) \in \{-k/2, k/2\}$. For any vertex set A , take $\alpha(A)$ to be a rational number such that $\alpha(A) \in \{0, \pm 1/2, \dots, \pm k/2\}$ and $\alpha(A) \stackrel{k}{\equiv} p(A) - d_G(A)/2$, where $p(A) = \sum_{v \in A} p(v) - e_G(A)$ and $d_G(A) = \sum_{v \in A} d_G(v) - 2e_G(A)$. Now, we present some basic properties of α in the following propositions.

Proposition 3 *For any two vertex sets A and B of the graph G , the following results hold*

1. *If $\alpha(A) \stackrel{k}{\equiv} \pm \alpha(B)$, then $|\alpha(A)| = |\alpha(B)|$.*
2. *If $A \cap B = \emptyset$, then $\alpha(A \cup B) \stackrel{k}{\equiv} \alpha(A) + \alpha(B)$.*
3. *$|\alpha(A)| = |\alpha(A^c)|$.*
4. *If $\alpha(v_0) = 0$ for a vertex v_0 with $v_0 \in V(G) \setminus A$, then $|\alpha(A)| = |\alpha(A \cup \{v_0\})|$.*

5. If $d_G(A) \geq 3k - 3$, then $d_G(A) \geq (2k - 2) + 2|\alpha(A)|$.

6. $d_G(A) - 2|\alpha(A)|$ is an even integer.

Proof. To obtain (1), one can conclude that $|\alpha(A) \mp \alpha(B)| \in \{0, k\}$ which implies that $|\alpha(A)| = |\alpha(B)|$. To prove (2), it suffices to check that

$$\begin{aligned} \alpha(A) + \alpha(B) &\stackrel{k}{\equiv} (p(A) - d_G(A)/2) + (p(B) - d_G(B)/2) \\ &\stackrel{k}{\equiv} (p(A) + p(B) - e_G(A, B)) - (d_G(A) + d_G(B) - 2e_G(A, B))/2 \\ &\stackrel{k}{\equiv} p(A \cup B) - d_G(A \cup B)/2 \\ &\stackrel{k}{\equiv} \alpha(A \cup B). \end{aligned}$$

Moreover, $\alpha(V(G)) = 0$, since

$$p(V(G)) \stackrel{k}{\equiv} \sum_{v \in V(G)} p(v) - e_G(V(G)) \stackrel{k}{\equiv} \sum_{v \in V(G)} p(v) - |E(G)| \stackrel{k}{\equiv} 0.$$

Hence $\alpha(A) + \alpha(A^c) \stackrel{k}{\equiv} 0$ and $|\alpha(A)| = |\alpha(A^c)|$ which establishes (3). The proof of (4) can be obtained from $\alpha(A) \stackrel{k}{\equiv} \alpha(A) + \alpha(v_0) \stackrel{k}{\equiv} \alpha(A \cup \{v_0\})$. Since $\alpha(A) + d_G(A)/2$ is an integer, $2\alpha(A) + d_G(A)$ is even and so $d_G(A) - 2|\alpha(A)|$ is even which implies (6). Note that $|\alpha(A)| \leq k/2$. If $|\alpha(A)| = k/2$, then $d_G(A)$ and k have the same parity. Since $3k - 3$ and k have different parity, we have $|\alpha(A)| < k/2$, when $d_G(A) = 3k - 3$. This can complete the proof. \square

Proposition 4 *If G' is a graph obtained from G by lifting two edges xu and yu , then for every vertex set A we have $|\alpha'(A)| = |\alpha(A)|$ where*

$$p' = \begin{cases} p - \chi_u - \chi_x, & \text{if } xu \text{ and } yu \text{ are parallel;} \\ p - \chi_u, & \text{if } xu \text{ and } yu \text{ are not parallel.} \end{cases}$$

Now, we are ready to refine the main result in [20].

Theorem 8. *Let G be a graph with $z_0 \in V(G)$, let k be an integer, $k \geq 3$, and let $p : V(G) \rightarrow \mathbb{Z}_k$ be a mapping with $|E(G)| \stackrel{k}{\equiv} \sum_{v \in V(G)} p(v)$. Let D_{z_0} be a pre-orientation of $E(z_0)$ that is the set of edges incident with z_0 . Let $V_0 = \{v \in V(G) - z_0 : \alpha(v) = 0\}$. If $V_0 \neq \emptyset$, we let v_0 be a vertex of V_0 with smallest degree. Assume that*

(i) $d_G(z_0) \leq 2k - 2 + 2|\alpha(z_0)|$, and the edges incident with z_0 are pre-directed such that $d_G^+(z_0) \stackrel{k}{\equiv} p(z_0)$.

(ii) $d_G(A) \geq 2k - 2 + 2|\alpha(A)|$, for any vertex set A with $\emptyset \subsetneq A \subsetneq V(G) \setminus z_0$ and $A \neq \{v_0\}$.

Then the pre-orientation D_{z_0} can be extended to a p -orientation D of G such that for each vertex v ,

$$|d_G^+(v) - d_G(v)/2| \leq k - 1 + |\alpha(v)|.$$

Proof. The proof is by contradiction. We assume (reductio ad absurdum) that (G, p, z_0) is a counterexample so that $|V(G)| \geq 3$. That is, the graph G with mapping p satisfies the conditions of the theorem but some pre-orientation D_{z_0} cannot be extended to a p -orientation of G with the desired properties. Let \mathcal{M} be the collection of counterexamples (G, p, z_0) such that $|V(G)| + |E(G - z_0)|$ is minimum. The proof is divided into two parts. The first part, Claims 1-5 below, establishes some properties of all members of \mathcal{M} . In the second part we choose a member (G, p, z_0) of \mathcal{M} such that $|E(G)|$ is minimum and prove that it is not a counterexample, yielding a contradiction. If we work with distinct graphs G, G' , we use the terms $p(A)$ and $\alpha(A)$ when A is a vertex set of G , and $p'(A)$ and $\alpha'(A)$ when A is a vertex set of G' .

Part I. Some properties of \mathcal{M} .

In Part 1 we let (G, p, z_0) be any member of \mathcal{M} .

Claim 1. For every vertex set $A \subsetneq V(G) \setminus z_0$ with $|A| \geq 2$, we have $d_G(A) \geq 2k + 2|\alpha(A)|$.

If $d_G(A) < 2k + 2|\alpha(A)|$, then we first get an extension of D_{z_0} to the contracted graph $H = G/A$ by the minimality property of G , since $|V(H)| < |V(G)|$ and $|E(H - z_0)| \leq |E(G - z_0)|$. Then all edges of the edge-cut $[A, A^c]$ are oriented in this extension, where $A^c = V(G) \setminus A$ and $[A, A^c]$ is the set of edges with exactly one end in A . Similarly we then contract A^c into a single vertex as a new z_0 , and again, we use the minimality of G to extend the orientation of $[A, A^c]$ to the edges of G with both ends in A . \square

Claim 2. $V_0 = \emptyset$.

Suppose $V_0 \neq \emptyset$ and v_0 is a vertex of V_0 with smallest degree. We can assume that $d_G(v_0) \geq 2$. Otherwise v_0 is an isolated vertex and we can remove it and use the minimality of G . If v_0 has at least two neighbours, we lift one pair of edges incident with v_0 which are not parallel. Claim 1 implies that the resulting graph G' with the modified mapping $p' = p - \chi_{v_0}$ satisfies the hypotheses of the theorem. Since $|E(G' - z_0)| < |E(G - z_0)|$, it holds that G' has the desired orientation, and so does G , a contradiction.

Now suppose v_0 has only one neighbour x . We must have $x \neq z_0$. Otherwise,

$$|\alpha(W)| = |\alpha(\{z_0, v_0\})| = |\alpha(z_0)|,$$

where $W = V(G) \setminus \{z_0, v_0\}$, and then

$$d_G(z_0) = d_G(W) + d_G(v_0) \geq (2k - 2) + 2|\alpha(W)| + 2 = 2k + 2|\alpha(z_0)|.$$

a contradiction to condition (i). If $|V(G)| = 3$, then we extend D_{z_0} to an orientation of G by orienting half of the edges between x and v_0 toward v_0 and the other half away from v_0 , yielding a contradiction. For the case $|V(G)| > 3$, we have

$$d_G(x) = d_G(\{x, v_0\}) + d_G(v_0) \geq (2k - 2) + 2|\alpha(\{x, v_0\})| + 2 = 2k + 2|\alpha(x)|.$$

Then, we lift one pair of edges incident with v_0 and x which are parallel. Claim 1 implies that the resulting graph G' with the modified mapping $p' = p - \chi_x - \chi_{v_0}$ satisfies the hypotheses of the theorem. Since $|E(G' - z_0)| < |E(G - z_0)|$, it holds that G' has the desired orientation, and so does G , again a contradiction. \square

Claim 3. $G - z_0$ is connected, and $d_G(z_0) \geq k$

Suppose $G - z_0$ is disconnected and let U and W be two components of $G - z_0$. By condition (ii) and Claim 2, we have $d_G(U) \geq 2k - 2$ and $d_G(W) \geq 2k - 2$. Then

$$d_G(z_0) \geq d_G(U) + d_G(W) > (2k - 2) + k \geq (2k - 2) + 2|\alpha(z_0)|,$$

a contradiction to condition (i).

Suppose $d_G(z_0) \leq k - 1$ and let G' be the graph constructed from G by replacing an edge xy of $G - z_0$ with a directed path of length two through z_0 with $p' = p + \chi_{z_0}$. We have $d_{G'}(z_0) \leq (k - 1) + 2 \leq 2k - 2 \leq (2k - 2) + 2|\alpha'(z_0)|$ and hence G' satisfies condition (i). For any vertex set A described in condition (ii), $d_{G'}(A) = d_G(A) + 2$, if A contains both x and y , and $d_{G'}(A) = d_G(A)$ otherwise. So condition (ii) is clearly satisfied. Since $|V(G')| = |V(G)|$ and $|E(G' - z_0)| < |E(G - z_0)|$, this implies (by the definition of \mathcal{M}) that an extension of D_{z_0} exists in G' . This orientation results in an orientation of G , a contradiction. \square

Claim 3.A. For each vertex $v \in V(G) - z_0$, $d_G(v) = 2k - 2 + 2|\alpha(v)|$.

Suppose otherwise that $d_G(x) \geq 2k + 2|\alpha(x)|$ for a vertex x with $x \neq z_0$. First, assume that x has at least two neighbours. Then we lift one pair of edges incident with x which are not parallel. Claim 1 implies that the resulting graph G' with the modified mapping $p' = p - \chi_x$ satisfies the hypotheses of the theorem. Since $|E(G' - z_0)| < |E(G - z_0)|$, it holds that G' has the desired orientation, and so does G , a contradiction. Next, assume that x has only one neighbour y . By Claim 3, we must have $y \neq z_0$ and so

$$d_G(y) \geq \begin{cases} d_G(x) + d_G(z_0) \geq 2k + k \geq 2k + 2|\alpha(y)|, & \text{if } |V(G)| = 3; \\ d_G(x) + d_G(\{x, y\}) \geq 2k + 2k - 2 \geq 2k + 2|\alpha(y)|, & \text{if } |V(G)| > 3. \end{cases}$$

Then, we lift one pair of edges incident with x and y which are parallel. Claim 1 implies that the resulting graph G' with the modified mapping $p' = p - \chi_x - \chi_y$ satisfies the hypotheses of the theorem. Since $|E(G' - z_0)| < |E(G - z_0)|$, it holds that G' has the desired orientation, and so does G , again a contradiction. \square

By condition (i) and Claim 3.A, if G has a p -orientation, then for each vertex v the following condition automatically holds,

$$|d_G^+(v) - d_G(v)/2| \leq k - 1 + |\alpha(v)|.$$

Claim 4. For any two distinct vertices $x, y \in V(G) - z_0$, we have $\alpha(x)\alpha(y) > 0$.

Suppose $\alpha(x)\alpha(y) \leq 0$. By Claim 2, we may assume that $\alpha(x) > 0$ and $\alpha(y) < 0$. By Claim 3, since $G - z_0$ is connected, we may also assume that $xy \in E(G - z_0)$. Let $G' = G - xy$, and take $p' = p - \chi_x$ to be the modified mapping.

Then $|V(G')| = |V(G)|$ and $|E(G' - z_0)| < |E(G - z_0)|$. If G' and p' satisfy the conditions of the theorem, then by the definition of \mathcal{M} , the pre-orientation can be extended to a p' -orientation of G' and further to a p -orientation of G by adding a directed edge from x to y , yielding a contradiction. Hence, it suffices to verify the conditions of the theorem for G' and p' . Moreover, we only need to verify condition (ii) for single vertices x and y and vertex sets A such that $|A| \geq 2$ and $d_{G'}(A) = d_G(A) - 1$ which are affected by the deletion of xy .

Condition (ii) is satisfied for x and y , since

$$\begin{aligned} d_{G'}(x) &= d_G(x) - 1, & p'(x) &= p(x) - 1, & \alpha'(x) &= \alpha(x) - 1/2, & |\alpha'(x)| &= |\alpha(x)| - 1/2, \\ d_{G'}(y) &= d_G(y) - 1, & p'(y) &= p(y), & \alpha'(y) &= \alpha(y) + 1/2, & |\alpha'(y)| &= |\alpha(y)| - 1/2. \end{aligned}$$

For any vertex set A (in condition (ii)) such that $|A| \geq 2$ and $d_{G'}(A) = d_G(A) - 1$, we have $|\alpha'(A)| = |\alpha(A) \pm 1/2| \leq |\alpha(A)| + 1/2$ and by Claim 1,

$$d_{G'}(A) = d_G(A) - 1 \geq (2k + 2|\alpha(A)|) - 1 \geq (2k - 2) + 2|\alpha'(A)|.$$

Hence condition (ii) is verified for A . So $\alpha(x)\alpha(y) > 0$. \square

Let $V^+ = \{x \in V(G) - z_0 : 0 < \alpha(x) < k/2\}$ and $V^- = \{x \in V(G) - z_0 : -k/2 < \alpha(x) < 0\}$.

Note that if $k/2 \stackrel{k}{\equiv} p(x) - d_G(x)/2$, then $\alpha(x)$ has two possible values, namely $k/2$ and $-k/2$.

Claim 5 . $V(G) - z_0 = V^+$ or $V(G) - z_0 = V^-$.

By Claim 4, we have $V^+ = \emptyset$ or $V^- = \emptyset$. So it suffices to prove that $|\alpha(x)| < k/2$ for any vertex x other than z_0 . If $x \in V(G) - z_0$ such that $|\alpha(x)| = k/2$, then for any vertex y distinct from x and z_0 , we can choose $\alpha(x) = k/2$ or $\alpha(x) = -k/2$ such that $\alpha(x)\alpha(y) \leq 0$ and get a contradiction to Claim 4. \square

Part II. Minimum members of \mathcal{M} .

Now choose (G, p, z_0) to be a member of \mathcal{M} such that $|E(G)|$ is minimum. Without loss of generality, assume that $V(G) - z_0 = V^+$. For if $V(G) - z_0 = V^-$, we reverse the directions of all edges

incident with z_0 and replace $p(x)$ by $d_G(x) - p(x)$ for each vertex x (including z_0). Then the resulting graph with the modified mapping satisfies $V(G) - z_0 = V^+$ and is also a minimum member of \mathcal{M} .

For each vertex $x \in V(G) - z_0$,

$$d_G(x) \geq (2k - 2) + 2\alpha(x) \text{ and } 0 < \alpha(x) < k/2.$$

Claim 6. $d_G(z_0) = k + p(z_0)$, and all edges incident with z_0 are directed away from z_0 .

By Claim 3, z_0 has a neighbour x . By Claim 5, $0 < \alpha(x) < k/2$. If xz_0 is directed toward z_0 , then we delete xz_0 . By a proof similar to that of Claim 4, the resulting graph with modified mapping $p' = p - \chi_x$ satisfies the conditions of the theorem. Since $|V(G')| = |V(G)|$ and $|E(G')| < |E(G)|$, and (G, p, z_0) is a smallest member of \mathcal{M} , the pre-orientation can be extended to a p' -orientation of G' and then to a p -orientation of G which contradicts the fact that (G, p, z_0) is a counterexample. So all edges incident with z_0 are directed away from z_0 , and $d_G(z_0) = d_G^+(z_0) \stackrel{k}{\equiv} p(z_0)$. Now, we can assume that $d_G^+(z_0) = d_G(z_0) = sk + p(z_0)$, where $s \geq 0$. By condition (i), we have $d_G(z_0) \leq (2k - 2) + 2|\alpha(z_0)| \leq 3k - 2$ and so $s \leq 2$. In the case $s = 2$, we derive that $p(z_0)/2 < |\alpha(z_0)|$. Since

$$p(z_0) \stackrel{k}{\equiv} \alpha(z_0) + d_G(z_0)/2 \stackrel{k}{\equiv} \alpha(z_0) + (2k + p(z_0))/2 \stackrel{k}{\equiv} \alpha(z_0) + p(z_0)/2,$$

we also derive that $\alpha(z_0) \stackrel{k}{\equiv} p(z_0)/2$ which is a contradiction. By Claim 3, we have $d_G(z_0) = k + p(z_0)$. \square

The final step: (G, p, z_0) is not a counterexample.

By Claim 6, let x be a neighbour of z_0 , and let e be an edge directed from z_0 to x . We replace e by $k - 1$ multiple directed edges from x to z_0 . Let G' be the resulting graph with $p' = p - \chi_x - \chi_{z_0}$. We are going to prove that G' with the mapping p' satisfies all conditions of the theorem and, furthermore, $-k/2 < \alpha'(x) < 0$ for the vertex x . By Claim 6, $d_G(z_0) = k + p(z_0)$. Since $p'(z_0) = p(z_0) - 1$ and $d_{G'}(z_0) = d_G(z_0) + k - 2$, we have

$$\alpha'(z_0) \stackrel{k}{\equiv} p'(z_0) - d_{G'}(z_0)/2 \stackrel{k}{\equiv} p(z_0) - 1 - (d_G(z_0) + k - 2)/2 \stackrel{k}{\equiv} p(z_0)/2.$$

This implies that $2|\alpha'(z_0)| = p(z_0)$ and therefore, $d_{G'}(z_0) = (2k - 2) + 2|\alpha'(z_0)|$. So, condition (i) is satisfied for G' and p' .

For condition (ii), we only need to consider x and vertex sets containing x . Since $|\alpha(x)| \geq 1/2$, we have

$$d_{G'}(x) = d_G(x) + (k - 2) \geq (2k - 2) + 2|\alpha(x)| + (k - 2) \geq 3k - 3,$$

and hence $d_{G'}(x) \geq (2k - 2) + 2|\alpha'(x)|$. In addition,

$$\alpha'(x) \stackrel{k}{\equiv} p'(x) - d_{G'}(x)/2 \stackrel{k}{\equiv} p(x) - 1 - (d_G(x) + (k - 2))/2 \stackrel{k}{\equiv} \alpha(x) - k/2.$$

Since $0 < \alpha(x) < k/2$, we have $\alpha'(x) = \alpha(x) - k/2$ and so $-k/2 < \alpha'(x) < 0$. By Claim 1, for any non-trivial vertex set A of G described in condition (ii) and containing x , we also have

$$d_{G'}(A) = d_G(A) + k - 2 \geq 3k - 2 = (2k - 2) + k \geq (2k - 2) + 2|\alpha'(A)|.$$

So, condition (ii) is also satisfied.

Now if (G', p', z_0) is also a counterexample, then $(G', p', z_0) \in \mathcal{M}$, since $|V(G')| = |V(G)|$ and $|E(G' - z_0)| = |E(G - z_0)|$. But we have $V'^+ = V(G') - \{z_0, x\}$ and $V'^- = \{x\}$, a contradiction to Claim 5. So (G', p', z_0) is not a counterexample, and hence G' has a p' -orientation. Then the corresponding orientation of G (obtained by replacing the $k - 1$ edges from x to z_0 with one edge in opposite direction) is a p -orientation of G satisfying the theorem. This completes the proof. \square

Remark 2. Note that the condition $|d_G^+(v) - d_G(v)/2| \leq k - 1 + |\alpha(v)|$ directly implies that $|d_G^+(v) - d_G(v)/2| < k$, and on the other hand $\lfloor \frac{d_G(v)}{2} \rfloor - (k - 1) \leq d_G^+(v) \leq \lceil \frac{d_G(v)}{2} \rceil + (k - 1)$, because $d_G^+(v) - d_G(v)/2 \in \{\pm|\alpha(v)|, \pm(k - |\alpha(v)|)\}$ and also $d_G^+(v) - d_G(v)/2 = 0$ when $\alpha(v) = 0$.

When z_0 has not small enough degree, one can replace the following version of Theorem 8. Note that by ignoring the extra condition on z_0 , the proof can easily be obtained after adding an additional vertex of degree zero which plays the role of the vertex z_0 in Theorem 8.

Corollary 1. *Let G be a graph, let k be an integer, $k \geq 3$, and let $p : V(G) \rightarrow \mathbb{Z}_k$ be a mapping with $|E(G)| \equiv \sum_{v \in V(G)} p(v) \pmod k$. If for every vertex set A with $\emptyset \subsetneq A \subsetneq V(G)$, $d_G(A) \geq 2k - 2 + 2|\alpha(A)|$, then G has a p -orientation such that for each vertex v ,*

$$|d_G^+(v) - d_G(v)/2| \leq k - 1 + |\alpha(v)|.$$

Furthermore, for an arbitrary vertex z_0 , $d_G^+(z_0)$ can be assigned to any plausible integer value in whose interval.

Proof. The proof is by induction on $|V(G)| + |E(G)|$. For $|V(G)| \leq 2$, the proof is trivial. Hence we may assume that $|V(G)| \geq 3$. If $d_G(z_0) = 2k - 2 + 2|\alpha(z_0)|$, then the proof can easily be derived from Theorem 8. So, suppose $d_G(z_0) \geq 2k + 2|\alpha(z_0)|$. We claim that for any vertex set $A \subsetneq V(G) \setminus z_0$ with $|A| \geq 2$, we have $d_G(A) \geq 2k + 2|\alpha(A)|$ and so $d_G(A^c) \geq 2k + 2|\alpha(A^c)|$. For, if $d_G(A) < 2k + 2|\alpha(A)|$, then by a proof similar to that of Claim 1, we apply induction to G/A and then we apply Theorem 8 to G/A^c . If z_0 has at least two neighbours, then we lift one non-parallel pair of edges incident with z_0 . Otherwise, if z_0 has only one neighbour y , we lift one parallel pair of edges incident with z_0 and y . In this case, we have

$$d_G(y) \geq d_G(z_0) + d_G(\{z_0, y\}) \geq 2k + 2k - 2 \geq 2k + 2|\alpha(y)|.$$

By applying the induction hypothesis the proof can be completed. \square

The following corollary partially answer to Conjecture 1.

Corollary 2. *Let G be a graph, let k be an integer, $k \geq 3$, let $p : V(G) \rightarrow Z_k$ be a mapping with $|E(G)| \equiv \sum_{v \in V(G)} p(v) \pmod k$. If G is $(3k - 3)$ -edge-connected, then G has a p -orientation such that for each vertex v ,*

$$\lfloor \frac{d_G(v)}{2} \rfloor - (k - 1) \leq d_G^+(v) \leq \lceil \frac{d_G(v)}{2} \rceil + (k - 1).$$

Furthermore, for an arbitrary vertex z_0 , $d_G^+(z_0)$ can be assigned to any plausible integer value in whose interval.

Remark 3. Let G be a $(4k - 1)$ -edge-connected $(4k - 1)$ -regular graph of size divisible by k . It is easy to verify that G has no orientation with out-degrees in S , for any $S = \{2k, 3k\}, \{k - 1, 2k - 1\}$. This shows that the lower bound in Corollary 2 cannot be increased up to $\lceil \frac{d_G(v)}{2} \rceil - (k - 1)$ and the upper bound in Corollary 2 cannot be decreased down to $\lfloor \frac{d_G(v)}{2} \rfloor + (k - 1)$.

Jaeger [17] conjectured that every $(2k - 2)$ -edge-connected graph G with odd positive integer k has an orientation such that for each vertex v , $d_G^+(v) \equiv d_G^-(v) \pmod k$. By Propositions 1 and 2, this conjecture could be formulated to a much simpler and stronger version by restricting out-degrees. Now, this version can partially be confirmed as the following immediate conclusion.

Corollary 3. *Let G be a graph and let k be an odd positive integer. If G is $(3k - 3)$ -edge-connected, then G has an orientation such that for each vertex v ,*

$$d_G^+(v) \in \left\{ \frac{d_G(v)}{2} - \frac{k}{2}, \frac{d_G(v)}{2}, \frac{d_G(v)}{2} + \frac{k}{2} \right\}.$$

We also conclude the following conclusion concerning non-Eulerian orientations of Eulerian graphs.

Corollary 4. *Let G be a graph of even order with even degrees and let k be a positive integer. If G is $(6k - 2)$ -edge-connected, then G has an orientation such that for each vertex v ,*

$$d_G^+(v) \in \left\{ \frac{d_G(v)}{2} - k, \frac{d_G(v)}{2} + k \right\}.$$

It would be interesting to determine the sharp edge-connectivity of Corollary 4. Motivated by the special case $k = 1$, we pose the following question.

Question 1 *Let G be a graph of even order with even degrees and let k be a positive integer. Is it true that if G is $(4k - 2)$ -edge-connected, then G has an orientation such that for each vertex v ,*

$$d_G^+(v) \in \left\{ \frac{d_G(v)}{2} - k, \frac{d_G(v)}{2} + k \right\}?$$

2.5 Replacing odd-edge-connectivity condition

Motivated by Theorem 4.12 in [20], we improve Theorem 8 as the following strengthened version which discounts the condition $d_G(A) \geq 2k - 2 + 2|\alpha(A)|$ for any vertex set A with $\alpha(A) = 0$.

Theorem 9. *Let G be a graph with $z_0 \in V(G)$, let k be an integer, $k \geq 3$, and let $p : V(G) \rightarrow Z_k$ be a mapping with $|E(G)| \stackrel{k}{\equiv} \sum_{v \in V(G)} p(v)$. Let D_{z_0} be a pre-orientation of $E(z_0)$ that is the set of edges incident with z_0 . Assume that*

- (i) $\alpha(z_0) \neq 0$.
- (ii) $d_G(z_0) \leq 2k - 2 + 2|\alpha(z_0)|$, and the edges incident with z_0 are pre-directed such that $d_G^+(z_0) \stackrel{k}{\equiv} p(z_0)$.
- (iii) $d_G(A) \geq 2k - 2 + 2|\alpha(A)|$, for any vertex set A with $\emptyset \subsetneq A \subsetneq V(G) \setminus z_0$ and $\alpha(A) \neq 0$.

Then the pre-orientation D_{z_0} can be extended to a p -orientation D of G such that for each vertex v ,

$$|d_G^+(v) - d_G(v)/2| \leq k - 1 + |\alpha(v)|.$$

Proof. The proof is by induction on $|V(G)| + |E(G)|$. For $|V(G)| \leq 3$, the assertion holds by Theorem 8. So, suppose $|V(G)| \geq 4$. We claim that for any vertex set $A \subsetneq V(G) \setminus z_0$ such that $\alpha(A) \neq 0$ and $|A| \geq 2$, we have $d_G(A) \geq 2k + 2|\alpha(A)|$. For, if $d_G(A) < 2k + 2|\alpha(A)|$, then by a proof similar to that of Claim 1, we apply induction to G/A and then to G/A^c . Then by a proof similar to that of Claim 2, we claim that there is no vertex v_0 of G such that $\alpha(v_0) = 0$; for otherwise we either remove v_0 or lift one pair of edges incident with v_0 , and next we apply induction. Now G must have a vertex set $A \subsetneq V(G) \setminus z_0$ such that $\alpha(A) = 0$, $|A| \geq 2$, and $d_G(A) \leq 2k - 2$. For otherwise G satisfies the conditions of Theorem 8, and Theorem 9 follows. Choose A with minimal $|A|$. We contract A and use induction. Then we contract A^c and by the minimality of A we can apply Theorem 8 to the graph G/A^c . \square

When z_0 has not small enough degree, one can replace the following version of Theorem 9.

Corollary 5. *Let G be a graph, let k be an integer, $k \geq 3$, and let $p : V(G) \rightarrow Z_k$ be a mapping with $|E(G)| \stackrel{k}{\equiv} \sum_{v \in V(G)} p(v)$. If for every vertex set A with $\alpha(A) \neq 0$, $d_G(A) \geq 2k - 2 + 2|\alpha(A)|$, then G has a p -orientation such that for each vertex v ,*

$$|d_G^+(v) - d_G(v)/2| \leq k - 1 + |\alpha(v)|.$$

Furthermore, for an arbitrary vertex z_0 , $d_G^+(z_0)$ can be assigned to any plausible integer value in whose interval.

Proof. By induction on $|V(G)| + |E(G)|$. For $|V(G)| \leq 2$, the proof is trivial. Hence we may assume that $|V(G)| \geq 3$. Note that if $\alpha = 0$, then the graph G whose degrees are even and the theorem clearly holds. Also if $\alpha(z_0) = 0$, then we can take another vertex as z_0 without this property. Hence we may assume that $\alpha(z_0) \neq 0$. If $d_G(z_0) = 2k - 2 + 2|\alpha(z_0)|$, then the conclusion trivially holds, using Theorem 9. So, suppose $d_G(z_0) \geq 2k + 2|\alpha(z_0)|$. We claim that for any vertex set $A \subsetneq V(G) \setminus z_0$ such that $\alpha(A) \neq 0$ and $|A| \geq 2$, we have $d_G(A) \geq 2k + 2|\alpha(A)|$ and so $d_G(A^c) \geq 2k + 2|\alpha(A^c)|$. For, if $d_G(A) < 2k + 2|\alpha(A)|$, then by a proof similar to that of Claim 1, we apply induction to G/A and then we apply Theorem 9 to G/A^c . If z_0 has at least two neighbours, then we lift one non-parallel pair of edges incident with z_0 . Otherwise, if z_0 has only one neighbour y , we lift one parallel pair of edges incident with z_0 and y . In this case, we have

$$d_G(y) \geq d_G(z_0) + d_G(\{z_0, y\}) \geq \begin{cases} 2k + 2|\alpha(z_0)| = 2k + 2|\alpha(y)|, & \text{if } \alpha(\{z_0, y\}) = 0; \\ 2k + 2k - 2 \geq 2k + 2|\alpha(y)|, & \text{if } \alpha(\{z_0, y\}) \neq 0. \end{cases}$$

By applying the induction hypothesis the proof can be completed. \square

A revised version of Jaeger's Conjecture [17], which was proposed by Zhang [36], says that every $(2k-1)$ -odd-edge-connected graph G with odd positive integer k has an orientation such that for each vertex v , $d_G^+(v) \stackrel{k}{\equiv} d_G^-(v)$. By Lemma 2.2 in [36], this conjecture could similarly be formulated to a much simpler and stronger version by restricting out-degrees. This version can partially be confirmed as the following immediate conclusion.

Corollary 6. *Let G be a graph and let k be an odd positive integer. If G is $(3k-2)$ -odd-edge-connected, then G has an orientation such that for each vertex v ,*

$$d_G^+(v) \in \left\{ \frac{d_G(v)}{2} - \frac{k}{2}, \frac{d_G(v)}{2}, \frac{d_G(v)}{2} + \frac{k}{2} \right\}.$$

Corollary 7. *Let G be a graph of even order with even degrees and let k be a positive integer. If for every vertex set A of odd elements, $d_G(A) \geq 6k - 2$, then G has an orientation such that for each vertex v ,*

$$d_G^+(v) \in \left\{ \frac{d_G(v)}{2} - k, \frac{d_G(v)}{2} + k \right\}.$$

Proof. Apply Corollary 5 with $p(v) = d_G(v)/2 - k \pmod{2k}$, where $k \geq 2$. Also, use the fact that for every vertex set A of even elements, we have $\alpha(A) = 0$. For the special case $k = 1$, we can replace the condition $d_G(A) \geq 2$ for every vertex set A of odd elements. For this purpose, we need to apply Theorem 6 to each component of G separately. \square

Question 2 *Let G be a graph of even order with even degrees and let k be a positive integer. Is it true that if for every vertex set A of odd elements, $d_G(A) \geq 4k - 2$, then G has an orientation such that for each vertex v ,*

$$d_G^+(v) \in \left\{ \frac{d_G(v)}{2} - k, \frac{d_G(v)}{2} + k \right\}?$$

2.6 Lifting operations and preserving parity edge-connectivity

In this subsection, we provide a stronger version for Lemma 2.2 in [36] and furthermore Propositions 1 and 2, by combining the proof's ideas in [13, 36]. This version allows us to preserve odd-edge-connectivity and even-edge-connectivity simultaneously.

Theorem 10. *Let G be a graph with the vertex u and let m and m' be two nonnegative integers with $m' \geq m$. Assume that $d_G(u)$ is even or $d_G(u) \geq 2m' + 2$. If for all vertex sets A with $\emptyset \subsetneq A \subsetneq V(G) \setminus u$,*

$$d_G(A) \geq \begin{cases} 2m, & \text{when } d_G(A) \text{ is even;} \\ 2m' + 1, & \text{when } d_G(A) \text{ is odd,} \end{cases}$$

then for any edge xu incident with u there is another edge yu incident with u such that by lifting them the resulting graph H satisfies the following condition for all vertex sets A with $\emptyset \subsetneq A \subsetneq V(H) \setminus u$,

$$d_H(A) \geq \begin{cases} 2m, & \text{when } d_H(A) \text{ is even;} \\ 2m' + 1, & \text{when } d_H(A) \text{ is odd.} \end{cases}$$

Furthermore, if the vertex u has at least two neighbours, then we can have $y \neq x$.

Proof. For $|V(G)| = 2$, there is nothing to prove. So, suppose $|V(G)| \geq 3$. Let yu be an edge incident with u distinct from xu . Suppose the theorem is false. Thus there is a vertex set Y with $x, y \in Y \subsetneq V(G) \setminus u$ such that $d_G(Y) = 2m' + 1$ or $d_G(Y) = 2m$. Consider Y with maximum $|Y|$. First we claim that there is an edge zu incident with u such that $z \notin Y$. Otherwise, all neighbours of u lie in Y . Thus $d_G(Y) = d_G(Y^c \setminus u) + d_G(u)$. Since $d_G(Y) \leq 2m' + 1$, we must have $d_G(u) \leq 2m' + 1$ and hence $d_G(u)$ is even. In this case, $d_G(Y)$ and $d_G(Y^c \setminus u)$ have the same parity and so $d_G(Y^c \setminus u) \geq 2m = d_G(Y)$, when $d_G(Y)$ is even, and $d_G(Y^c \setminus u) \geq 2m' + 1 = d_G(Y)$, when $d_G(Y)$ is odd. Notice that $\emptyset \subsetneq Y^c \setminus u \subsetneq V(G) \setminus u$. Since $d_G(u) > 0$, we arrive at a contradiction and the claim holds. Since u has at least two neighbours, we may assume that $y \neq x$. Corresponding to the edge zu , again there is a vertex set Z with $x, z \in Z \subsetneq V(G) \setminus u$ such that $d_G(Z) = 2m' + 1$ or $d_G(Z) = 2m$. Maximality property of Y implies that $Y \cap Z^c \neq \emptyset$. Note also that $x \in Y \cap Z$ and $u \in Y^c \cap Z^c$. If $d_G(Y)$ and $d_G(Z)$ have different parity, then $d_G(Y \cap Z^c)$ and $d_G(Z \cap Y^c)$ also have different parity and so

$$\begin{aligned} (2m' + 2m + 1) &= d_G(Y) + d_G(Z) \\ &= d_G(Y \cap Z^c) + d_G(Z \cap Y^c) + 2e_G(Y \cap Z, Y^c \cap Z^c) \geq (2m' + 2m + 1) + 2, \end{aligned}$$

which is impossible. Hence $d_G(Y)$ and $d_G(Z)$ have the same parity and also $d_G(Y \cap Z^c)$ and $d_G(Z \cap Y^c)$ have the same parity. Since $m' \geq m$, we must have

$$\begin{aligned} 2m' + 1 + 2m' + 1 &= d_G(Y) + d_G(Z) \\ &= d_G(Y \cap Z^c) + d_G(Z \cap Y^c) + 2e_G(Y \cap Z, Y^c \cap Z^c) \geq 2m + 2m + 2. \end{aligned}$$

Thus $d_G(Y)$ and $d_G(Z)$ are odd, and also $d_G(Y \cap Z^c)$ and $d_G(Z \cap Y^c)$ are even. This implies that $d_G(Y \cup Z)$ is odd and also $d_G(Y \cap Z)$, because of $d_G(Y \cup Z) = d_G(Y) + d_G(Z \cap Y^c) - 2e_G(Y, Z \cap Y^c)$. If $d_G(u) \geq 2m' + 2$, then $d_G(Y \cup Z) \geq 2m' + 1$ whether $Y \cup Z \subsetneq V(G) \setminus u$ or not. Also, if $d_G(u)$ is even, then $Y \cup Z \subsetneq V(G) \setminus u$. Since $Y \cap Z \neq \emptyset$, we have

$$\begin{aligned} 2m' + 1 + 2m' + 1 &= d_G(Y) + d_G(Z) \\ &= d_G(Y \cap Z) + d_G(Y \cup Z) + 2e(Y \cap Z^c, Z \cap Y^c) \geq 2m' + 1 + 2m' + 1 \end{aligned}$$

Thus $d_G(Y \cup Z) = 2m' + 1 = d_G(Y \cap Z)$. This implies that $Y \cup Z \neq V(G) \setminus u$. Since $Y \cup Z \supsetneq Y$, we arrive at a contradiction, as desired. \square

Let m and m' be two nonnegative integers. We say that a graph G is $(2m' + 1, 2m)$ -edge-connected, if for every nonempty proper vertex set A , we have $d_G(A) \geq 2m' + 1$ when $d_G(A)$ is odd and $d_G(A) \geq 2m$ when $d_G(A)$ is even. Note that every $2m$ -edge-connected graph is $(2m + 1, 2m)$ -edge-connected. Now, we form the following results with respect to this definition.

Corollary 8. *Let G be a $(2m' + 1, 2m)$ -edge-connected graph with $m' \geq m \geq 0$. If $u \in V(G)$ and $d_G(u) \geq 2m' + 2$, then there are two edges incident with u such that by lifting them the resulting graph is still $(2m' + 1, 2m)$ -edge-connected.*

Corollary 9. *Let G be a $(2m' + 1, 2m)$ -edge-connected with $m' \geq m \geq 0$. If $u \in V(G)$ and $d_G(u)$ is even, then we can alternatively lift $d_G(u)/2$ disjoint pair of edges incident with u such that the resulting graph H with $V(H) = V(G) \setminus \{u\}$ is still $(2m' + 1, 2m)$ -edge-connected.*

Motivated by Corollary 7, we also formulate the following theorem which preserves a new edge-connectivity in Eulerian graphs.

Theorem 11. *Let G be a graph with even degrees. Let u be a vertex and let n and n' be two integers with $n' \geq n \geq 0$ and with $d_G(u) \geq 2n' + 2$. If for all vertex sets A with $\emptyset \subsetneq A \subsetneq V(G) \setminus u$,*

$$d_G(A) \geq \begin{cases} 2n, & \text{when } |A| \text{ is even;} \\ 2n', & \text{when } |A| \text{ is odd,} \end{cases}$$

then for any edge xu incident with u there is another edge yu incident with u such that by lifting them the resulting graph H satisfies the following condition for all vertex sets A with $\emptyset \subsetneq A \subsetneq V(H) \setminus u$,

$$d_H(A) \geq \begin{cases} 2n, & \text{when } |A| \text{ is even;} \\ 2n', & \text{when } |A| \text{ is odd.} \end{cases}$$

Furthermore, if the vertex u has at least two neighbours, then we can have $y \neq x$.

Proof. For $|V(G)| = 2$, there is nothing to prove. So, suppose $|V(G)| \geq 3$. Let yu be an edge incident with u distinct from xu . Suppose the theorem is false. Thus there is a vertex set Y with $x, y \in Y \subsetneq V(G) \setminus u$ such that $d_G(Y) = 2n'$ or $d_G(Y) = 2n$. Consider Y with maximum $|Y|$. First we claim that there is an edge zu incident with u such that $z \notin Y$. Otherwise, all neighbours of u lie in Y . Thus $d_G(Y) = d_G(Y^c \setminus u) + d_G(u)$. Since $d_G(u) \geq 2n' + 2$, we arrive at a contradiction and the claim easily holds. Since u has at least two neighbours, we may assume that $y \neq x$. Corresponding to the edge zu , again there is a vertex set Z with $x, z \in Z \subsetneq V(G) \setminus u$ such that $d_G(Z) = 2n'$ or $d_G(Z) = 2n$. Maximality property of Y implies that $Y \cap Z^c \neq \emptyset$. Note also that $x \in Y \cap Z$ and $u \in Y^c \cap Z^c$. If $|Y|$ and $|Z|$ have different parity, then $|Y \cap Z^c|$ and $|Z \cap Y^c|$ also have different parity and so

$$\begin{aligned} 2n' + 2n &= d_G(Y) + d_G(Z) \\ &= d_G(Y \cap Z^c) + d_G(Z \cap Y^c) + 2e_G(Y \cap Z, Y^c \cap Z^c) \geq 2n' + 2n + 2, \end{aligned}$$

which is impossible. Hence $|Y|$ and $|Z|$ have the same parity and so $|Y \cap Z^c|$ and $|Z \cap Y^c|$ have the same parity. Since $n' \geq n$, we must have

$$\begin{aligned} 2n' + 2n' &= d_G(Y) + d_G(Z) \\ &= d_G(Y \cap Z^c) + d_G(Z \cap Y^c) + 2e_G(Y \cap Z, Y^c \cap Z^c) \geq 2n + 2n + 2. \end{aligned}$$

Thus $|Y|$ and $|Z|$ are odd and also $|Y \cap Z^c|$ and $|Z \cap Y^c|$ are even. This implies that $|Y \cup Z|$ and $|Y \cap Z|$ are odd, because of $|Y \cup Z| + |Y \cap Z| = |Y| + |Z|$. Also, the condition $d_G(u) \geq 2n' + 2$ conclude that $d_G(Y \cup Z) \geq 2n'$, whether $Y \cup Z \subsetneq V(G) \setminus u$ or not. Since $Y \cap Z \neq \emptyset$, we have

$$\begin{aligned} 2n' + 2n' &= d_G(Y) + d_G(Z) \\ &= d_G(Y \cap Z) + d_G(Y \cup Z) + 2e(Y \cap Z^c, Z \cap Y^c) \geq 2n' + 2n' \end{aligned}$$

Thus $d_G(Y \cup Z) = 2n' = d_G(Y \cap Z)$. This implies that $Y \cup Z \neq V(G) \setminus u$. Since $Y \cup Z \supsetneq Y$, we arrive at a contradiction, as desired. \square

Remark 4. Let S be a set of pair of edges incident with u as $\{e_i, e_j\}$ and let Q be a graph with vertices of all edges incident with u and with the edge set S . The same arguments in the proof of Theorems 10 and 11 can imply that if the graph Q is connected, then there are two edges e_i and e_j incident with u such that $\{e_i, e_j\} \in S$ and by lifting them the resulting graph satisfies the desired properties. To see this, it suffices to deduce that when Q is connected there are two edges e and e' with $\{e, e'\} \in S$ which are separated by the set Y and next form the set Z to arrive a contradiction. For the case that Q is a path, this fact is an useful result for finding lifting operations on surfaces which simultaneously preserve parity edge-connectivity and embedding property, see Section 2 in [36].

2.7 Lifting operation and preserving tree-connectivity

In this subsection, we present a sufficient condition for the existence of lifting operations which preserves tree-connectivity. In the meantime, we establish the following simple but useful lemma.

Lemma 2. *Let G be a connected graph with $|V(G)| \geq 2$ and with $u \in V(G)$. If $d_G(u) \geq 2\omega(G \setminus u) - 2$, then there are some non-parallel pair of edges incident with u such that by lifting them the resulting graph H with $V(H) = V(G) \setminus u$ is still connected, where $\omega(G \setminus u)$ denotes the number of components of $G \setminus u$.*

Proof. For the case $\omega(G \setminus u) = 1$, there is nothing to prove. For the case $\omega(G \setminus u) = 2$, let $\{xu, yu\}$ be a pair of edges incident with u such that x and y lie in different components of $G \setminus u$. Next, lift them and call the resulting graph G' and set $H = G' \setminus u$. Now, suppose $\omega(G \setminus u) \geq 3$. Since $d_G(u) \geq 2\omega(G \setminus u) - 2 \geq \omega(G \setminus u) + 1$, there is a component C of $G \setminus u$ and two edges xu and zu such that $x, z \in V(C)$. Since G is connected, there is an edge yu incident with u such that $y \notin V(C)$. Lift xu and yu and call the resulting graph G_1 . According to construction, G_1 is connected, $\omega(G_1 \setminus u) = \omega(G \setminus u) - 1$, and $d_{G_1}(u) = d_G(u) - 2$. On the other hand G_1 satisfies the lemma. By repeating this process, we derive connected graphs G_1, G_2, \dots, G_t which G_{i+1} is obtained from G_i and also $\omega(G_t \setminus u) = 2$. Finally, lift two edges of G_t incident with u such that $G_{t+1} \setminus u$ is connected, where G_{t+1} is the resulting graph. Now, it is enough to set $H = G_{t+1} \setminus u$. \square

Now, we are in a position to prove the main result of this subsection.

Theorem 12. *Let G be an m -tree-connected graph with $u \in V(G)$. If $d_G(u) \leq 2m$, then there are at most $(d_G(u) - m)$ non-parallel pair of edges incident with u such that by lifting them the resulting graph H with $V(H) = V(G) \setminus u$ is still m -tree-connected.*

Proof. Assume that T_1, \dots, T_n and $T'_1, \dots, T'_{n'}$ are m edge-disjoint spanning trees of G such that $d_{T_i}(u) \geq 2$ and $d_{T'_j}(u) = 1$. Let e_j be the unique edge of T'_j incident with u . Since $d_G(u) \leq 2m$, we have $n' \geq \sum_{1 \leq i \leq n} (d_{T_i}(u) - 2)$, and so one can take E_1, \dots, E_n to be n disjoint subsets of $\{e_1, \dots, e_{n'}\}$ such that $|E_i| = d_{T_i}(u) - 2$. Define $\mathcal{T}_i = T_i + E_i$ so that

$$d_{\mathcal{T}_i}(u) = 2d_{T_i}(u) - 2 = 2\omega(T_i \setminus u) - 2 \geq 2\omega(\mathcal{T}_i \setminus u) - 2.$$

By Lemma 2, there are some non-parallel pair of edges of \mathcal{T}_i incident with u such that by lifting them the resulting graph H_i with $V(H_i) = V(\mathcal{T}_i) \setminus u$ is still connected. Now, in G lift the same pair of edges. Since every $T'_j \setminus u$ is connected, the resulting graph H with $V(H) = V(G) \setminus u$ is still m -tree-connected. Note the number of lifted pairs in G is at most $\frac{1}{2} \sum_{1 \leq i \leq n} d_{\mathcal{T}_i}(u)$ and the following inequality can complete the proof.

$$\frac{1}{2} \sum_{1 \leq i \leq n} d_{\mathcal{T}_i}(u) = \frac{1}{2} \sum_{1 \leq i \leq n} (2d_{T_i}(u) - 2) \leq d_G(u) - m.$$

□

2.8 Graphs with tree-connectivity at least $2k - 2$

In this subsection, we improve the needed edge-connectivity in Corollary 2, but require the graph to have many edge-disjoint spanning trees.

Theorem 13. *Let G be a graph with $|V(G)| \geq 2$, let k be an integer, $k \geq 3$, and let $p : V(G) \rightarrow \mathbb{Z}_k$ be a mapping with $|E(G)| \equiv \sum_{v \in V(G)} p(v) \pmod{k}$. If G is $(2k - 2)$ -tree-connected, then G has a p -orientation such that for each vertex v ,*

$$k/2 - 1 \leq d_G^+(v) \leq d_G(v) - k/2 + 1.$$

Proof. The proof is by induction on $|V(G)|$. For $|V(G)| = 2$ the proof is straightforward. So suppose $|V(G)| \geq 3$. We first claim that for every vertex u , $d_G(u) > 3k - 3$. For, if $d_G(u) \leq 3k - 3$, then by Theorem 12 there are at most $d_G(u) - 2k + 2$ non-parallel pair of edges incident with u such that by lifting them the resulting graph H with $V(H) = V(G) \setminus u$ is still $(2k - 2)$ -tree-connected. Let Q be the set of edges incident with u that are not lifted. Assume that t pair of edges are lifted. Direct t_1 edges of Q away from u and t_2 remaining edges toward u , where $t_1 + t_2 \equiv p(u) \pmod{k}$. Since $|Q| \geq 4k - 4 - d_G(u) \geq k - 1$, t_1 is well-defined and since $d_G(u) \geq 2k - 2$, we can have $t_1 + t \geq k/2 - 1$ and $t_2 + t \geq k/2 - 1$. Now, for each vertex v of H , define $p'(v) = p(v) - q(v)$, where $q(v)$ is number of edges in Q directed away from v . It is easy to check that $|E(H)| \equiv \sum_{v \in V(H)} p'(v) \pmod{k}$. By the induction hypothesis, H has a p' -orientation modulo k such that for each $v \in V(H)$, $d_H^+(v) \geq k/2 - 1$ and $d_H^-(v) \geq k/2 - 1$. This orientation induces a p -orientation for G such that for each vertex v with $v \neq u$, $d_G^+(v) \geq d_H^+(v)$ and $d_G^-(v) \geq d_H^-(v)$. In particular, for the vertex u , we have $d_G^+(u) = t_1 + t \geq k/2 - 1$ and $d_G^-(u) = t_2 + t \geq k/2 - 1$. This completes the claim's proof. Now G must have a vertex set A such that $|A| \geq 2$, $|A^c| \geq 2$, and $d_G(A) < 2k - 2 + 2|\alpha(A)|$. For otherwise G satisfies the conditions of Corollary 1 and G has a p -orientation such that for each vertex v , $\lfloor k/2 \rfloor \leq \lfloor d_G(v)/2 \rfloor - (k - 1) \leq d_G^+(v) \leq \lceil d_G(v)/2 \rceil + (k - 1) \leq d_G(v) - \lfloor k/2 \rfloor$, because of $\delta(G) \geq 3k - 2$. Choose A with minimal $|A|$. We contract A and use induction. Notice that G/A is also $(2k - 2)$ -tree-connected. Then we contract A^c and by the minimality of A we can apply Theorem 8 to the graph G/A^c . □

The next corollary can be proved similarly to Proposition 2 in [31] and was similarly appeared in [5, 6, 23]. This result reduces the edge-connectivity needed to decompose a graph into isomorphic copies of a fixed tree in [2, 5, 6, 23, 28, 29, 31]. For instance, it can reduce the required edge-connectivity of Theorem 4.2 in [5] down to 32 with exactly the same proof.

Corollary 10. *Let G be a bipartite graph with the bipartition (V_1, V_2) . Let k and m be two nonnegative integers with $k \geq 3$. If G is a $(2m + 2k - 2)$ -tree-connected and $|E(G)|$ is divisible by k , then G admits*

a decomposition into two m -tree-connected factors G_1 and G_2 such that for each $v \in V_i$, $d_{G_i}(v)$ is divisible by k and $d_{G_i}(v) \geq m + k/2 - 1$.

The idea of lifting operations on vertices with degrees bounded by $3k - 2$ and $4k - 4$ can help us to establish the following complicated but stronger version of Theorem 13. This result is useful when one needs to have large out-degrees (except for a single vertex).

Theorem 14. *Let G be a graph with $|V(G)| \geq 2$ and with $z_0 \in V(G)$, let k be an integer, $k \geq 3$, and let $p : V(G) \rightarrow \mathbb{Z}_k$ be a mapping with $|E(G)| \equiv \sum_{v \in V(G)} p(v) \pmod{k}$. For each vertex v , take $s(v)$ to be an integer with $0 \leq s(v) \leq k - 1$. If G is $(2k - 2)$ -tree-connected, then G has a p -orientation such that for each $v \in V(G) \setminus z_0$,*

$$s(v) \leq d_G^+(v) \leq d_G(v) - k + 1 + s(v).$$

Furthermore, we can have $k - 1 - \max\{s(v) : v \in V(G) \setminus z_0\} \leq d_G^+(z_0) \leq d_G(z_0) - \min\{s(v) : v \in V(G) \setminus z_0\}$.

Proof. The proof is by induction on $|V(G)|$. For $|V(G)| = 2$ the is straightforward. So suppose $|V(G)| \geq 3$. We first claim that for every vertex u with $u \neq z_0$, we have $d_G(u) > 4k - 4 - 2|\alpha(u)|$. For, if $d_G(u) \leq (4k - 4) - 2|\alpha(u)|$, then by Theorem 12 there are at most $d_G(u) - (2k - 2)$ non-parallel pair of edges incident with u such that by lifting them the resulting graph H with $V(H) = V(G) \setminus u$ is still $(2k - 2)$ -tree-connected. Let Q be the set of edges incident with u that are not lifted. Assume that t pair of edges are lifted. Direct t_1 edges of Q away from u and t_2 remaining edges toward u . For the case $d_G(u) \geq 3k - 2$, since $|Q| \geq d_G(u) - 2(d_G(u) - (2k - 2)) \geq 4k - 4 - d_G(u) \geq |2\alpha(u)|$, t_1 and t_2 can be selected such that $t_1 + t = d_G(u)/2 + \alpha(u)$ and $t_2 + t = d_G(u)/2 - \alpha(u)$, and so $t_1 + t \geq k - 1 \geq s(u)$ and $t_2 + t \geq k - 1 \geq k - 1 - s(u)$. For the case $d_G(u) \leq 3k - 3$, since $|Q| \geq 4k - 4 - d_G(u) \geq k - 1$ and $d_G(u) \geq 2k - 2$, again t_1 and t_2 can be selected such that $t_1 + t \equiv^k p(u)$, $d_G^+(u) \geq s(u)$ and $d_G^-(u) \geq k - 1 - s(u)$. Now, for each vertex v of H , define $p'(v) = p(v) - q(v)$, where $q(v)$ is number of edges in Q directed away from v . It is easy to check that $|E(H)| \equiv \sum_{v \in V(H)} p'(v) \pmod{k}$. By the induction hypothesis, H has a p' -orientation modulo k such that for each $v \in V(H) \setminus z_0$, $d_H^+(v) \geq s(v)$, $d_H^-(v) \geq k - 1 - s(v)$, and also $d_H^+(z_0) \geq k - 1 - \max\{s(v) : v \in V(H) \setminus z_0\}$ and $d_H^-(z_0) \geq \min\{s(v) : v \in V(H) \setminus z_0\}$. This orientation induces a p -orientation for G such that for each vertex v with $v \neq u$, $d_G^+(v) \geq d_H^+(v)$ and $d_G^-(v) \geq d_H^-(v)$. In particular, for the vertex u , we have $d_G^+(u) = t_1 + t \geq s(u)$ and $d_G^-(u) = t_2 + t \geq k - 1 - s(u)$. This completes the claim's proof. Now G must have a vertex set A such that $A \subsetneq V(G) \setminus z_0$, $|A| \geq 2$, and $d_G(A) < 2k - 2 + 2|\alpha(A)|$. For otherwise G satisfies the conditions of Theorem 8 or Corollary 1 and so G has a p -orientation such that for each vertex v ,

$$d_G^+(v) \in \begin{cases} \{d_G(v)/2 + \alpha(v), d_G(v)/2 + \alpha(v) - k\}, & \text{if } \alpha(v) > 0; \\ \{d_G(v)/2 + \alpha(v), d_G(v)/2 + \alpha(v) + k\}, & \text{if } \alpha(v) < 0; \\ \{d_G(v)/2\}, & \text{if } \alpha(v) = 0. \end{cases}$$

If $d_G(z_0) \geq 2k - 2 + 2|\alpha(z_0)|$, then we can have $d_G^+(z_0) = d_G(z_0)/2 + \alpha(z_0)$ and so $k - 1 \leq d_G^+(z_0) \leq d_G(z_0) - k + 1$. Also if $d_G(z_0) < 2k - 2 + 2|\alpha(z_0)|$, then in order to apply Theorem 8 the edges of G incident with z_0 can be directed such that $d_G^+(z_0) \stackrel{k}{=} p(z_0)$ and $k - 1 - \max\{s(v) : v \in V(G) \setminus z_0\} \leq d_G^+(z_0) \leq d_G(z_0) - \min\{s(v) : v \in V(G) \setminus z_0\}$, because of $d_G(z_0) \geq 2k - 2$. Note also that the condition $d_G(v) > 4k - 4 - 2|\alpha(v)|$, in both cases, implies that $k - 1 \leq d_G^+(v) \leq d_G(v) - k + 1$. Choose A with minimal $|A|$. We contract A and use induction. Notice that G/A is also $(2k - 2)$ -tree-connected. Then we contract A^c and by the minimality of A we can apply Theorem 8 to the graph G/A^c . \square

Remark 5. Assume that, in Theorem 14, for each $v \in V(G) \setminus z_0$, we have $d_G(v) \geq 2k - 2 + 2|\alpha(v)|$. Also, suppose that (i) $s(v) \leq d_G(v)/2 + \alpha(v) - k$ and $k - 1 - s(v) \leq d_G(v)/2 - \alpha(v)$ when $\alpha(v) > 0$, and (ii) $s(v) \leq d_G(v)/2 + \alpha(v)$ and $k - 1 - s(v) \leq d_G(v)/2 - \alpha(v) - k$ when $\alpha(v) < 0$. In this case, one can replace the condition $s(z_0) \leq d_G^+(z_0) \leq d_G(z_0) - k + 1 + s(z_0)$ for the vertex z_0 . To see this, it suffices to apply Theorem 8 or Corollary 1, or otherwise for applying Theorem 14, in contracted graph G/A , the contraction of A plays the role of z_0 .

We here propose the following interesting conjecture which is a stronger version of Conjecture 2 in [15] due to Barát, Gerbner, and Thomassé [2, Conjecture 6]. Note that if the following conjecture would be true, then Conjecture 1 can be confirmed by replacing edge-connectedness $2k$, using Propositions 1 and 2 and Lemma 9.

Conjecture 2. Let G be a graph with $|V(G)| \geq 2$ and with $z_0 \in V(G)$. Let k be a positive integer and let $p : V(G) \rightarrow \mathbb{Z}_k$ be a mapping with $|E(G)| \stackrel{k}{=} \sum_{v \in V(G)} p(v)$. For each vertex v , take $s(v)$ to be an integer with $s(v) \in \{0, 1\}$. If G is k -tree-connected, then G has a p -orientation such that for each $v \in V(G) \setminus z_0$,

$$s(v) \leq d_G^+(v) \leq d_G(v) - 1 + s(v).$$

Furthermore, we can have $1 - \max\{s(v) : v \in V(G) \setminus z_0\} \leq d_G^+(z_0) \leq d_G(z_0) - \min\{s(v) : v \in V(G) \setminus z_0\}$.

Finally, we form the following theorem which a combination of it together with Theorem 13 can refine Theorem 7 in [12], see [32, Section 2].

Theorem 15. Every graph G of order n , $n \geq 2$, with at least $m(n - 1)$ edges contains an m -tree-connected subgraph with at least two vertices.

Proof. The proof is by induction on $|V(G)|$. For $|V(G)| = 2$, the proof is clear. So, suppose $|V(G)| \geq 3$. Suppose also the theorem is false. By Proposition 8, there exists a partition X_1, \dots, X_t of $V(G)$ such that $\sum_{1 \leq i \leq t} \frac{1}{2}d_G(X_i) < m(t - 1)$. By induction hypothesis, for every vertex set X_i , we have $e_G(X_i) \leq m(|X_i| - 1)$ whether $|X_i| = 1$ or not. Therefore,

$$m(n - 1) \leq |E(G)| = \sum_{1 \leq i \leq t} \left(\frac{1}{2}d_G(X_i) + e_G(X_i) \right) < m(t - 1) + m \sum_{1 \leq i \leq t} (|X_i| - 1) \leq m(n - 1).$$

This result is a contradiction, as desired. \square

2.9 Star-Decompositions

Barát and Thomassen (2006) conjectured that every 4-edge-connected 4-regular planar simple graph G of size divisible by 3 admits a claw-decomposition. Later, Lai (2007) disproves this conjecture by a class of 4-edge-connected 2-connected simple graphs. It remains to decide whether every essentially 6-edge-connected simple graph of size divisible by 3 with minimum degree at least 4 has a claw-decomposition. Surprisingly, the answer is false and we observe that the smallest such counterexamples have 21 vertices, see Figure 1.

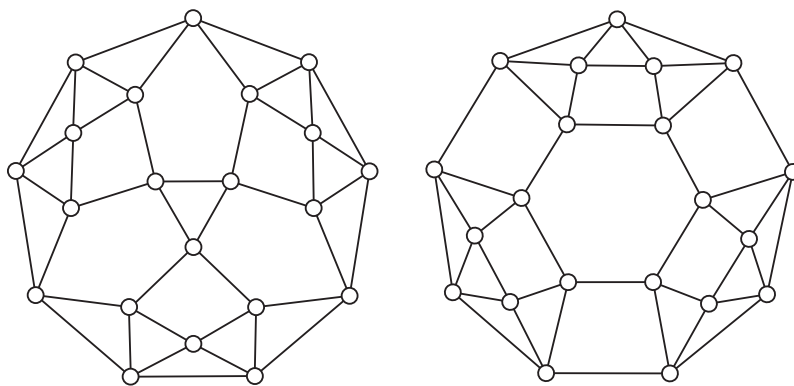


Figure 1: Two 4-regular planar simple graphs of order 21 without claw-decomposition.

By the following construction, we extend our search to 4-regular planar simple graphs of high enough order with the highest essentially edge-connectivity and with the highest vertex-connectivity. We also observe that the smallest such graphs with the described properties have 30 vertices, using a planar graph generator due to Brinkmann and McKay [7].

Theorem 16. *There are infinitely many 4-connected essentially 6-edge-connected 4-regular planar simple graphs of size divisible by 3 without claw-decomposition.*

Proof. Consider $3n$ copies of the graph in Figure 2 and for every $i \in \mathbb{Z}_{3n}$, add three edges $z_i z_{i+1}$, $x_i a_{i+1}$, and $y_i b_{i+1}$ to the new graph. Call the resulting graph G_{48n} . As observed in [3, 19], if a 4-regular graph G has a claw-decomposition, then the non-center vertices form an independent set of size $|V(G)|/3$. If G_{48n} has a claw-decomposition, then it must have an independent set X of size $16n$. But X has at most 5 vertices of every block and hence has at most $15n$ vertices which is a contradiction. The vertex connectivity and essentially edge-connectivity of G_{3n} can easily be verified. Figure 3 illustrates the graph G_{48} . \square

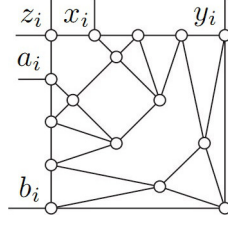


Figure 2: The block for constructing the family of graphs G_{48n} .

Note that the result of Lovász, Thomassen, Wu, and Zhang [20] implies that every essentially 6-edge-connected simple graph G of size divisible by 3 with minimum degree at least 5 admits a claw-decomposition, see [10, Theorem 1.1]. Motivated by Corollary 1, we form the following version which gives a bound for the number of claws with a fixed center.

Theorem 17. *Every essentially $(3k - 3)$ -edge-connected simple graph G of size divisible by k and with $\delta(G) \geq 2k - 1$ admits a k -star-decomposition such that every vertex v is the center of $\lfloor d_G(v)/2k \rfloor$ or $\lceil d_G(v)/2k \rceil$ stars.*

Proof. Define $p(v) = 0$, for each vertex v . Note that if $d_G(v) = 2k + r$ and $-1 \leq r \leq k - 4$, then $\alpha(v) \stackrel{k}{\equiv} -d_G(v)/2 \stackrel{k}{\equiv} -r/2$ and $-1/2 \leq r/2 \leq k/2 - 2$. This implies that $2|\alpha(v)| = |r|$ and $r + 2 \geq 2|\alpha(v)|$ and so $d_G(v) \geq 2k - 2 + 2|\alpha(v)|$. Therefore, by Corollary 1, the graph G has a p -orientation modulo k such that for each vertex v , $d_G(v)/2 - k < d_G^+(v) < d_G(v)/2 + k$. This implies that $d_G(v)/2k - 1 < d_G^+(v)/k < d_G(v)/2k + 1$. On the other hand $d_G^+(v)/k$ is equal to $\lfloor d_G(v)/2k \rfloor$ or $\lceil d_G(v)/2k \rceil$. Hence the theorem holds. \square

Remark 6. Note that for any $k \geq 4$, the Cartesian product of a cycle of order $3n$ and the complete graph of order $2k - 3$ is a $(2k - 2)$ -edge-connected $(2k - 2)$ -regular of size divisible by k with the highest essentially edge-connectivity $4k - 6$ and with the highest vertex-connectivity $2k - 2$, while has no k -star-decomposition. This refines a recent construction due to Delcourt and Postle [10] and the proof follows with exactly the same arguments. Note also that for a few number of vertices the lower bound in Theorem 17 can be replaced by $2k - 2$, by replacing Theorem 8 instead of Corollary 1.

3 Modulo factors with bounded degrees

In this section, we investigate highly edge-connected factors modulo k whose degrees are restricted to given small intervals. We begin with factors modulo 2 and deduce several corollaries specially for spanning Eulerian subgraphs.

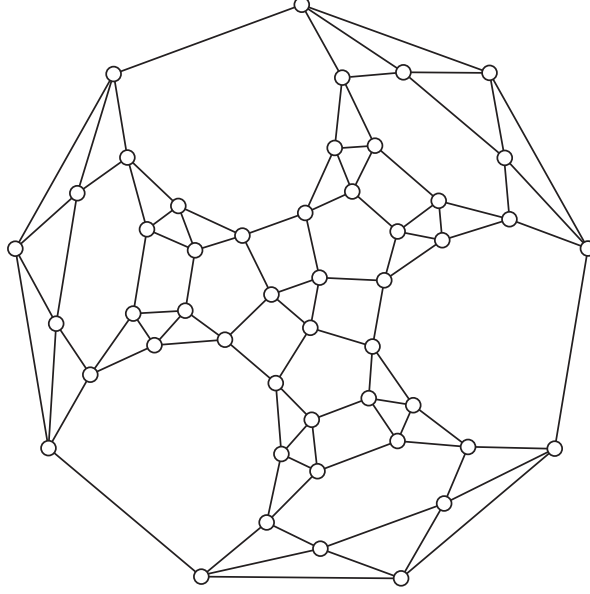


Figure 3: A 4-connected essentially 6-edge-connected 4-regular planar simple graph of order 48 without claw-decomposition.

3.1 Preliminaries

In this subsection, we state two propositions, for applying in the next subsections, and postpone their proofs until Section 4. The first proposition is involved by a parameter $s(v)$ which allow us to giving slightly different bounds. It is an useful tool for improving the bounds of some latter results. The proposition is also involved by an extra condition on $d_G(z_0)$ which can slightly improve some results in the subsequent subsection.

Proposition 5 *Let G be a $(2m_1 + 2m_2)$ -edge-connected graph. For each vertex v , take $s(v)$ to be an integer with $0 \leq s(v) \leq m_1$. If $m_2 \geq 1$, then G has two edge-disjoint factors M_1 and M_2 such that M_1 is m_1 -tree-connected, M_2 can be transformed into an m_2 -tree-connected graph L by alternatively lifting operations, and for each vertex v ,*

$$(i) \quad d_{M_1}(v) + \frac{d_{M_2}(v) - d_L(v)}{2} \geq \left\lfloor \frac{d_G(v)}{2} \right\rfloor - m_1 - m_2 + s(v),$$

$$(ii) \quad d_{M_1}(v) + \frac{d_{M_2}(v) + d_L(v)}{2} \leq \left\lceil \frac{d_G(v)}{2} \right\rceil + m_1 + m_2 + s(v).$$

Furthermore, for an arbitrary vertex z_0 , we can have $d_{M_1}(z_0) + \frac{d_{M_2}(z_0) + d_L(z_0)}{2} \leq \left\lfloor \frac{d_G(z_0)}{2} \right\rfloor + s(z_0)$, which $s(z_0) \leq m_1 + m_2$ when $d_G(z_0)$ is even and $s(z_0) \leq m_1 + m_2 + 1$ when $d_G(z_0)$ is odd.

Proposition 6 *Every $(2m_1 + 2m_2)$ -edge-connected graph G with $m_1 \geq 1$ has two edge-disjoint factors*

G_1 and G_2 such that G_1 is m_1 -tree-connected, G_2 consists of m_2 edge-disjoint spanning trees, and for each vertex v at least one of the following conditions holds,

- (i) $\lfloor \frac{d_G(v) - d_{G_2}(v)}{2} \rfloor \leq d_{G_1}(v) \leq \lceil \frac{d_G(v) - d_{G_2}(v)}{2} \rceil + m_1$,
- (ii) $\lfloor \frac{d_G(v)}{2} \rfloor - m_2 \leq d_{G_1}(v) \leq d_{G_1}(v) + d_{G_2}(v) \leq \lceil \frac{d_G(v)}{2} \rceil + m_1 + m_2$.

3.2 Factors modulo 2

In [21, 29, 34] the authors observed that edge-connectedness 1 is sufficient for a graph to have an f -factor modulo 2. This edge-connectedness cannot guarantee that degrees are strictly more than zero even in graphs with large degrees. Here, we show that edge-connectedness 2 is sufficient for a graph to have an f -factor modulo 2, where degrees fall in predetermined short intervals.

Theorem 18. *Let G be a graph and let $f : V(G) \rightarrow \mathbb{Z}_2$ be a mapping with $\sum_{v \in V(G)} f(v) \stackrel{2}{\equiv} 0$. If G is 2-edge-connected, then it has an f -factor H such that for each vertex v ,*

$$\lfloor \frac{d_G(v)}{2} \rfloor - 1 \leq d_H(v) \leq \lceil \frac{d_G(v)}{2} \rceil + 1.$$

Furthermore, for an arbitrary vertex z_0 , $d_H(z_0)$ can be assigned to any plausible integer value in whose interval.

We show also that edge-connectedness 4 is sufficient for a graph to have a connected f -factor modulo 2, where degrees fall in predetermined small intervals.

Theorem 19. *Let G be a graph and let $f : V(G) \rightarrow \mathbb{Z}_2$ be a mapping with $\sum_{v \in V(G)} f(v) \stackrel{2}{\equiv} 0$. If G is 4-edge-connected, then it has a connected f -factor H such that for each vertex v ,*

$$\lfloor \frac{d_G(v)}{2} \rfloor - l(v) \leq d_H(v) \leq \lceil \frac{d_G(v)}{2} \rceil + 2,$$

where $l(v) = 1$ when v has even degree, and $l(v) = 2$ when v has odd degree.

Corollary 11. *Every 4-edge-connected r -regular graph has a spanning Eulerian subgraph whose degrees lie in the set $\{4, 6\}$, for each $r = 8, 10$*

Corollary 12. *Every 4-edge-connected graph G has a spanning Eulerian subgraph H such that for each vertex v ,*

$$\lfloor \frac{d_G(v)}{2} \rfloor - 2 \leq d_H(v) \leq \lceil \frac{d_G(v)}{2} \rceil + 2.$$

Now, we want to prove the above-mentioned results as the following general version.

Theorem 20. Let G be a graph and let $f : V(G) \rightarrow \mathbb{Z}_2$ be a mapping with $\sum_{v \in V(G)} f(v) \stackrel{2}{\equiv} 0$. For each vertex v , take $s(v)$ to be an integer with $0 \leq s(v) \leq m$. If G is $(2m+2)$ -edge-connected, then it has an m -tree-connected f -factor H such that for each vertex v ,

$$\lfloor \frac{d_G(v)}{2} \rfloor - m - 1 + s(v) \leq d_H(v) \leq \lceil \frac{d_G(v)}{2} \rceil + m + 1 + s(v).$$

Furthermore, for an arbitrary vertex z_0 , we can have $d_H(v) \leq \lfloor \frac{d_G(z_0)}{2} \rfloor + s(z_0)$, which $s(z_0) \leq m+1$ when $d_G(z_0)$ is even and $s(z_0) \leq m+2$ when $d_G(z_0)$ is odd.

The following lemma brings us one step closer to proving Theorem 20. Note that if G is 2-edge-connected, then by an argument similar to the proof of Theorem 6, we could show that the orientation and factor of the following lemma can be found such that $d_{G_1}^+(v) + d_{F_2}(v)$ fall in the same interval stated in Theorem 18.

Lemma 3. Let G be a graph that decomposed into two factors G_1 and G_2 , and let $h : V(G) \rightarrow \mathbb{Z}_2$ be a mapping with $|E(G_1)| \stackrel{2}{\equiv} \sum_{v \in V(G)} h(v)$. If G is connected, then G_1 has an orientation and G_2 has a factor F_2 such that for each vertex v , $d_{G_1}^+(v) + d_{F_2}(v) \stackrel{2}{\equiv} h(v)$.

Proof. First, we prove the assertion for trees. By induction on $|V(G)|$. For $|V(G)| = 1$, there is nothing to prove. So, suppose $|V(G)| \geq 2$. Let x be a vertex of degree one and let xy be the unique edge incident with x . Take $G' = G - x$, $G'_1 = G_1 - x$, and $G'_2 = G_2 - x$. For each vertex $v \in V(G) \setminus \{x, y\}$, define $h'(v) = h(v)$. Also, define

$$h'(y) = \begin{cases} h(y) + h(x), & \text{if } xy \notin E(G_1); \\ h(y) + h(x) - 1, & \text{if } xy \in E(G_1). \end{cases}$$

It is easy to see that $|E(G'_1)| \stackrel{2}{\equiv} \sum_{v \in V(G')} h'(v)$. By the induction hypothesis, the graph G'_1 has an orientation and G'_2 has a factor F'_2 such that for each $v \in V(G')$, $d_{G'_1}^+(v) + d_{F'_2}(v) \stackrel{2}{\equiv} h'(v)$. Orient the edges of $E(G_1) \setminus \{xy\}$ similarly to $E(G'_1)$. For the case $xy \in E(G_1)$, orient the edge xy away from x , if $h(x) \stackrel{2}{\equiv} 1$; otherwise orient xy toward x . Let F_2 the factor of G_2 with the same edges of F'_2 and containing the edge xy , if $xy \notin E(G_1)$ and $h(x) \stackrel{2}{\equiv} 1$. This construction completes the proof for trees. If G is not tree, then first take a spanning tree T . Next, arbitrary orient the edges of $E(G_1) \setminus E(T)$ and arbitrary select some edges of $E(G_2) \setminus E(T)$ for making the factor of G_2 . Finally, extend them to G using the modified mapping and spanning tree T . \square

Using the above-mentioned lemma, we now prove the next lemma.

Lemma 4. Let G be a graph and let $f : V(G) \rightarrow \mathbb{Z}_2$ be a mapping with $\sum_{v \in V(G)} f(v) \stackrel{2}{\equiv} 0$. If G can be transformed into a connected graph L by alternatively lifting operations, then G has an f -factor H such that for each vertex v , $\frac{d_G(v) - d_L(v)}{2} \leq d_H(v) \leq \frac{d_G(v) + d_L(v)}{2}$.

Proof. For every edge xy in L , consider the trail P_{xy} in G with the end vertices x and y corresponding to xy . Let L_1 be the factor of L consisting of all edges xy such that P_{xy} has even size. Let L_2 be the factor of L consisting of all edges xy such that P_{xy} has odd size. Let R be the factor of G consisting the edges that lie out of such trails P_{xy} . Obviously, R whose degrees are even. Let Q_1, \dots, Q_t be the components of R of odd size and let $Q'_1, \dots, Q'_{t'}$ be the components of R of even size. For each i with $1 \leq i \leq t$, pick a vertex q_i of Q_i . For each vertex v of G , define

$$h(v) = \begin{cases} f(v) - \frac{d_G(v) - d_L(v)}{2}, & \text{if } v \notin \{q_1, \dots, q_t\}; \\ f(v) - \frac{d_G(v) - d_L(v)}{2} - 1, & \text{if } v \in \{q_1, \dots, q_t\}. \end{cases}$$

It is easy to see that $|E(L_1)| \stackrel{2}{=} |E(L)| - |E(L_2)| \stackrel{2}{=} |E(L)| - |E(G)| - t \stackrel{2}{=} \sum_{v \in V(L)} h(v)$. Lemma 3 implies that L_1 has an orientation and L_2 has a factor F_2 such that for each vertex v , $d_{L_1}^+(v) + d_{F_2}(v) \stackrel{2}{=} h(v)$. Note that for each vertex v , $d_L(v) \geq d_{L_1}^+(v) + d_{F_2}(v)$ and $d_L(v) \geq 1$. Now, one can color the edges of G red and blue using the following rules.

1. For every component Q'_i , alternatively color the edges by walking on an Eulerian tour.
2. For every component Q_i , alternatively color the edges by walking on an Eulerian tour, by starting at the vertex q_i which the initial color is blue if and only $d_{L_1}^+(q_i) + d_{F_2}(q_i) = 0$.
3. For every trail P_{xy} of even size, if the edge xy is directed away from x in L_1 , alternatively color the edges of P_{xy} , by walking on it by starting at x which the initial color is blue.
4. For every trail P_{xy} of odd size, alternatively color the edges of P_{xy} , by walking on it by starting at x which the initial color is blue if and only if $xy \in E(F_2)$.

Let H be the factor of G consisting of all edges having the blue color. According to the construction of H , for each vertex v , we have

$$d_H(v) = d_{L_1}^+(v) + d_{F_2}(v) + \begin{cases} \frac{d_G(v) - d_L(v)}{2}, & \text{if } v \notin \{q_1, \dots, q_t\}; \\ \frac{d_G(v) - d_L(v)}{2} + 1, & \text{if } v \in \{q_1, \dots, q_t\} \text{ and } d_{L_1}^+(v) + d_{F_2}(v) = 0; \\ \frac{d_G(v) - d_L(v)}{2} - 1, & \text{if } v \in \{q_1, \dots, q_t\} \text{ and } d_{L_1}^+(v) + d_{F_2}(v) \neq 0. \end{cases}$$

This implies that H is the desired f -factor. \square

Proof of Theorem 20. By Proposition 5, the graph G has two edge-disjoint factors M and M' such that M is m -tree-connected, M' can be transformed into a connected graph L by alternatively lifting operations, and for each vertex v , $d_M(v) + \frac{d_{M'}(v) - d_L(v)}{2} \geq \lfloor \frac{d_G(v)}{2} \rfloor - m - 1 + s(v)$ and $d_M(v) + \frac{d_{M'}(v) + d_L(v)}{2} \leq \lceil \frac{d_G(v)}{2} \rceil + m + 1 + s(v)$. For each vertex v , define $f'(v) = f(v) - d_M(v)$. Clearly, $\sum_{v \in V(M')} f'(v) \stackrel{2}{=} 0$. Lemma 4 implies that M' has an f' -factor H' such that for each vertex v , $\frac{d_{M'}(v) - d_L(v)}{2} \leq d_{H'}(v) \leq \frac{d_{M'}(v) + d_L(v)}{2}$. Consequently, $M \cup H'$ is an f -factor satisfying the desired property. Note that the extra condition on z_0 can easily be verified. \square

Proof of Theorem 19. Apply Theorem 20 for $m = 1$ by setting $s(v) = 1$ if and only if $d_G(v)$ is even and $d_G(v)/2 - 2 \stackrel{2}{\equiv} f(v)$. By considering that $d_G(v)/2 - 2 \not\stackrel{2}{\equiv} d_G(v)/2 + 3$, the theorem can easily be proved. \square

By replacing tree-connectivity condition $2m + 2$, Theorem 20 could be refined as the next theorem.

Theorem 21. *Let G be a graph and let $f : V(G) \rightarrow \mathbb{Z}_2$ be a mapping with $\sum_{v \in V(G)} f(v) \stackrel{2}{\equiv} 0$. If G is $(2m + 2)$ -tree-connected, then it has an m -tree-connected f -factor H such that for each vertex v ,*

$$\lfloor \frac{d_G(v)}{2} \rfloor - 1 \leq d_H(v) \leq \lceil \frac{d_G(v)}{2} \rceil + m + 1.$$

Proof. We have already observed that the assertion holds for $m = 0$, so assume that $m \geq 1$. Decompose G into two edge-disjoint factors G_1 and G_2 such that G_1 is $2m$ -tree-connected and G_2 is 2-tree-connected. Since G_2 has a spanning Eulerian subgraph [16], we may assume that G_2 is 2-edge-connected and whose degrees are even. Now, let H_1 be an m -tree-connected factor of G_1 such that for each vertex v , $\lfloor \frac{d_{G_1}(v)}{2} \rfloor \leq d_{H_1}(v) \leq \lceil \frac{d_{G_1}(v)}{2} \rceil + m$, using Theorem 8 in [1] (which is the special case $(m_1, m_2) = (m, 0)$ of Theorem 5). For each vertex v , define $f_2(v) = f(v) - d_{H_1}(v)$. Clearly, $\sum_{v \in V(G_2)} f_2(v) \stackrel{2}{\equiv} 0$. By Theorem 18, the graph G_2 has an f_2 -factor H_2 such that for each vertex v , $\frac{d_{G_2}(v)}{2} - 1 \leq d_{H_2}(v) \leq \frac{d_{G_2}(v)}{2} + 1$. It is easy to see that $H_1 \cup H_2$ is the desired f -factor. \square

Corollary 13. *Every 8-edge-connected 9-regular graph has a spanning Eulerian subgraph whose degrees lie in the set $\{4, 6\}$.*

Theorem 22. *Let G be a graph, and let $f : V(G) \rightarrow \mathbb{Z}_2$ be a mapping with $\sum_{v \in V(G)} f(v) \stackrel{2}{\equiv} 0$. If G is $(2m + 4)$ -edge-connected, then it has an m -tree-connected f -factor H such that for each vertex v ,*

$$\lfloor \frac{d_G(v)}{2} \rfloor - 2 \leq d_H(v) \leq \lceil \frac{d_G(v)}{2} \rceil + m + 2.$$

Proof. We have already observed that the assertion holds for $m = 0$, so assume that $m \geq 1$. By Proposition 6, the graph G has two edge-disjoint factors G_1 and G_2 such that G_1 is m -tree-connected, G_2 is 2-tree-connected, and for each vertex v , $\lfloor \frac{d_G(v) - d_{G_2}(v)}{2} \rfloor \leq d_{G_1}(v) \leq \lceil \frac{d_G(v) - d_{G_2}(v)}{2} \rceil + m$, or $\lfloor \frac{d_G(v)}{2} \rfloor - 2 \leq d_{G_1}(v) \leq d_{G_1}(v) + d_{G_2}(v) \leq \lceil \frac{d_G(v)}{2} \rceil + m + 2$. For each vertex v , define $f_2(v) = f(v) - d_{G_1}(v)$. Clearly, $\sum_{v \in V(G_2)} f_2(v) \stackrel{2}{\equiv} 0$. By Theorem 18, the graph G_2 has an f_2 -factor F_2 such that for each vertex v , $\lfloor \frac{d_{G_2}(v)}{2} \rfloor - 1 \leq d_{F_2}(v) \leq \lceil \frac{d_{G_2}(v)}{2} \rceil + 1$. It is not hard to check that $G_1 \cup F_2$ is the desired f -factor we are looking for. \square

3.3 Bipartite graphs

There is a one-to-one mapping between orientations and factors of any bipartite graph, which was utilized by Thomassen in [33] in order to establish Theorem 3. Using the same arguments, we derive

the following strengthened version.

Theorem 23. *Let G be a bipartite graph, let k be an integer, $k \geq 3$, and let $f : V(G) \rightarrow Z_k$ be a compatible mapping. If G is $(3k - 3)$ -edge-connected, then it has an f -factor H such that for each vertex v ,*

$$\lfloor \frac{d_G(v)}{2} \rfloor - (k - 1) \leq d_H(v) \leq \lceil \frac{d_G(v)}{2} \rceil + (k - 1).$$

Furthermore, for an arbitrary vertex z_0 , $d_H(z_0)$ can be assigned to any plausible integer value in whose interval.

Proof. Let (A, B) be a bipartition of G . For each $v \in A$, define $p(v) = f(v)$, and for each $v \in B$, define $p(v) = d_G(v) - f(v)$. By the assumption, we have $|E(G)| \stackrel{k}{\equiv} \sum_{v \in V(G)} p(v)$. Since G is $(3k - 3)$ -edge-connected, by Corollary 2, the graph G has a p -orientation modulo k such that for each vertex v , $\lfloor \frac{d_G(v)}{2} \rfloor - (k - 1) \leq d_G^+(v) \leq \lceil \frac{d_G(v)}{2} \rceil + (k - 1)$. Take H to be the factor of G consisting of all edges directed from A to B . Since for all vertices $v \in A$, $d_H(v) = d_G^+(v)$, and for all vertices $v \in B$, $d_H(v) = d_G(v) - d_G^+(v)$, the graph H is the desired f -factor. \square

We have the following immediate conclusions similar to Corollaries 3 and 4. Note the required edge-connectivity $3k - 3$ of the first one can be replaced by odd-edge-connectivity $3k - 2$ and the second one can be replaced by the condition $d_G(A) \geq 6k - 2$ for all vertex sets A of odd elements.

Corollary 14. *Let G be a bipartite graph and let k be an odd positive integer. If G is $(3k - 3)$ -edge-connected, then G has a factor H such that for each vertex v ,*

$$d_H(v) \in \left\{ \frac{d_G(v)}{2} - \frac{k}{2}, \frac{d_G(v)}{2}, \frac{d_G(v)}{2} + \frac{k}{2} \right\}.$$

Corollary 15. *Let G be a bipartite of even order with even degrees and let k be a positive integer. If G is $(6k - 2)$ -edge-connected, then G has a factor H such that for each vertex v ,*

$$d_H(v) \in \left\{ \frac{d_G(v)}{2} - k, \frac{d_G(v)}{2} + k \right\}.$$

The following theorem can be proved similarly to Theorem 23, using Theorem 13.

Theorem 24. *Let G be a bipartite graph with $|V(G)| \geq 2$, let k be an integer, $k \geq 3$, and let $f : V(G) \rightarrow Z_k$ be a compatible mapping. If G is $(2k - 2)$ -tree-connected, then it has an f -factor H such that for each vertex v ,*

$$k/2 - 1 \leq d_H(v) \leq d_G(v) - k/2 + 1.$$

Here, we generalize Theorems 23 and 24 for investigating highly edge-connected f -factors.

Theorem 25. *Let G be a bipartite graph, let k be an integer, $k \geq 3$, and let $f : V(G) \rightarrow \mathbb{Z}_k$ be a compatible mapping. For each vertex v , take $s(v)$ to be an integer with $0 \leq s(v) \leq m$. If G is $(2m + 4k - 4)$ -edge-connected, then it has an m -tree-connected f -factor H such that for each vertex v ,*

$$\lfloor \frac{d_G(v)}{2} \rfloor - 3k/2 + 1 - m + s(v) \leq d_H(v) \leq \lceil \frac{d_G(v)}{2} \rceil + 3k/2 - 1 + m + s(v).$$

Proof. For $|V(G)| = 1$ there is nothing to prove. So, suppose $|V(G)| \geq 2$. By Proposition 5, the graph G has two edge-disjoint factors M_1 and M_2 such that M_1 is m -tree-connected, M_2 can be transformed into a $(2k - 2)$ -tree-connected graph L by alternatively lifting operation, and for each vertex v ,

$$\begin{aligned} d_{M_1}(v) + \frac{d_{M_2}(v) - d_L(v)}{2} &\geq \lfloor \frac{d_G(v)}{2} \rfloor - (2k - 2) - m + s(v), \\ d_{M_1}(v) + \frac{d_{M_2}(v) + d_L(v)}{2} &\leq \lceil \frac{d_G(v)}{2} \rceil + (2k - 2) + m + s(v). \end{aligned}$$

For each vertex v , define

$$p(v) = \begin{cases} +f(v) - d_{M_1}(v) - \frac{d_{M_2}(v) - d_L(v)}{2}, & \text{if } v \in A; \\ -f(v) + d_{M_1}(v) + \frac{d_{M_2}(v) + d_L(v)}{2}, & \text{if } v \in B. \end{cases}$$

By the assumption, one can imply that $|E(L)| \equiv \sum_{v \in V(L)} p(v) \pmod{k}$. Hence by Theorem 13, the graph L has a p -orientation modulo k such that for each vertex v , $k/2 - 1 \leq d_L^+(v) \leq d_L(v) - k/2 + 1$. This orientation induces an orientation for M_2 such that for each vertex v , $d_{M_2}^+(v) = d_L^+(v) + \frac{d_{M_2}(v) - d_L(v)}{2}$. Let F_2 be the factor of M_2 consisting of all edges directed from A to B . For each vertex v , we have

$$d_{F_2}(v) = \begin{cases} d_{M_2}^+(v) = d_L^+(v) + \frac{d_{M_2}(v) - d_L(v)}{2}, & \text{if } v \in A; \\ d_{M_2}(v) - d_{M_2}^+(v) = -d_L^+(v) + \frac{d_{M_2}(v) + d_L(v)}{2}, & \text{if } v \in B. \end{cases}$$

It is not difficult to check that the graph $M_1 \cup F_2$ is the desired f -factor we are looking for. \square

Corollary 16. *Let G be a bipartite graph and let $f : V(G) \rightarrow \mathbb{Z}_3$ be a compatible mapping. If G is 10-edge-connected, then G has a connected f -factor H such that for each vertex v ,*

$$\lfloor \frac{d_G(v)}{2} \rfloor - l(v) \leq d_H(v) \leq \lceil \frac{d_G(v)}{2} \rceil + 4,$$

where $l(v) = 3$ when v has odd degree, and $l(v) = 4$ when v has even degree.

Here we propose the following interesting conjecture, which could easily be confirmed (using the proof's idea of Theorem 25), if Conjecture 2 would be true.

Conjecture 3. *Let G be a bipartite graph, let k and m be two positive integers, $k \geq 3$, and let $f : V(G) \rightarrow \mathbb{Z}_k$ be a compatible mapping. If G is $(2k + 2m)$ -edge-connected, then G has an m -tree-connected f -factor H such that for each vertex v ,*

$$\lfloor \frac{d_G(v)}{2} \rfloor - k - m + 1 \leq d_H(v) \leq \lceil \frac{d_G(v)}{2} \rceil + k + m.$$

When m is a large enough number compared to k , the lower bound of Theorem 25 can be improved to the following bound in compensation for greater edge-connectivity $2m + 6k - 6$.

Theorem 26. *Let G be a bipartite graph, let k be an integer, $k \geq 3$, and let $f : V(G) \rightarrow Z_k$ be a compatible mapping. If G is $(2m + 6k - 6)$ -edge-connected, then it has an m -tree-connected f -factor H such that for each vertex v ,*

$$\lfloor \frac{d_G(v)}{2} \rfloor - 5k/2 + 2 \leq d_H(v) \leq \lceil \frac{d_G(v)}{2} \rceil + 5k/2 - 2 + m.$$

Proof. We may assume that $|V(G)| \geq 2$ and $m \geq 1$. By Proposition 6, the graph G has two edge-disjoint factors G_1 and G_2 such that G_1 is m -tree-connected, G_2 is $(3k - 3)$ -edge-connected, and for each vertex v , $\lfloor \frac{d_G(v) - d_{G_2}(v)}{2} \rfloor \leq d_{G_1}(v) \leq \lceil \frac{d_G(v) - d_{G_2}(v)}{2} \rceil + m$ or $\lfloor \frac{d_G(v)}{2} \rfloor - (3k - 3) \leq d_{G_1}(v) \leq d_{G_1}(v) + d_{G_2}(v) \leq \lceil \frac{d_G(v)}{2} \rceil + m + (3k - 3)$. For each vertex v , define $f_2(v) = f(v) - d_{G_1}(v)$. Since f is compatible by G , the mapping f_2 is also compatible by G_2 . By Theorem 23, the graph G_2 has an f_2 -factor F_2 such that for each vertex v , $\lfloor \frac{d_{G_2}(v)}{2} \rfloor - (k - 1) \leq d_{F_2}(v) \leq \lceil \frac{d_{G_2}(v)}{2} \rceil + (k - 1)$ and so $k/2 - 1 \leq d_{F_2}(v) \leq d_{G_2}(v) - k/2 + 1$. It is not hard to check that $G_1 \cup F_2$ is the desired f -factor we are looking for. \square

By replacing tree-connectivity condition, we obtain the next theorem.

Theorem 27. *Let G be a bipartite graph, let k be an integer, $k \geq 3$, and let $f : V(G) \rightarrow Z_k$ be a compatible mapping. If G is $(2m + 3k - 3)$ -tree-connected, then it has an m -tree-connected f -factor H such that for each vertex v ,*

$$\lfloor \frac{d_G(v)}{2} \rfloor - (k - 1) \leq d_H(v) \leq \lceil \frac{d_G(v)}{2} \rceil + (k - 1) + m.$$

Proof. We may assume that $m \geq 1$. Decompose G into two edge-disjoint factors G'_1 and G'_2 such that G'_1 is $2m$ -tree-connected and G'_2 is $(3k - 3)$ -tree-connected. Let T_1, \dots, T_{2m} be $2m$ edge-disjoint spanning trees of G'_1 . Let F be a factor of T_1 such that $G'_1 - E(F)$ whose degrees are even, see Lemma 3. Set $G_1 = G'_1 - E(F)$ and $G_2 = G'_2 + E(F)$. Obviously, G_1 is $(2m - 1)$ -edge-connected and also G_2 is $(3k - 3)$ -edge-connected. Since G_1 is Eulerian, it is also $2m$ -edge-connected. Now, let H_1 be an m -tree-connected factor of G_1 such that for each vertex v , $\frac{d_{G_1}(v)}{2} \leq d_{H_1}(v) \leq \frac{d_{G_1}(v)}{2} + m$, using Theorem 8 in [1] (which is the special case $(m_1, m_2) = (m, 0)$ of Theorem 5). For each vertex v , define $f_2(v) = f(v) - d_{H_1}(v)$. Since f is compatible by G , the mapping f_2 is also compatible by G_2 . By Theorem 23, the graph G_2 has an f_2 -factor H_2 such that for each vertex v , $\lfloor \frac{d_{G_2}(v)}{2} \rfloor - (k - 1) \leq d_{H_2}(v) \leq \lceil \frac{d_{G_2}(v)}{2} \rceil + (k - 1)$. It is easy to check that $H_1 \cup H_2$ is the desired f -factor we are looking for. \square

3.4 Non-bipartite graphs

As we have seen for finding f -factors in bipartite graphs, the mapping f must be compatible. Surprisingly, when the main graph is non-bipartite with a large enough bipartite index, this condition can be removed, however more edge-connectivity is required. This fact recently discovered by Thomassen, Wu, and Zhang [34] for odd numbers k . Here, we improve their result as the following theorem.

Theorem 28. *Let G be a non-bipartite graph, let k an integer, $k \geq 3$, and let $f : V(G) \rightarrow Z_k$ be a mapping with the condition $\sum_{v \in V(G)} f(v) \stackrel{2}{\equiv} 0$ when k is even. Assume that $\text{bi}(G) \geq k - 1$ when k is odd and $\text{bi}(G) \geq k/2 - 1$ when k is even. If G is $(6k - 7)$ -edge-connected, then G has an f -factor H such that for each vertex v ,*

$$\begin{aligned} \lfloor \frac{d_G(v)}{2} \rfloor - k - k/4 + 1/4 &\leq d_H(v) \leq \lceil \frac{d_G(v)}{2} \rceil + k + k/4 - 1/4 && \text{if } k \text{ is even;} \\ \lfloor \frac{d_G(v)}{2} \rfloor - 3k/2 &\leq d_H(v) \leq \lceil \frac{d_G(v)}{2} \rceil + 3k/2, && \text{if } k \text{ is odd.} \end{aligned}$$

Before to start we recall an implicit idea from [34] and develop it for even numbers k .

Lemma 5. *Let G be a graph, let k an integer, $k \geq 3$, and let $f : V(G) \rightarrow Z_k$ be a mapping with the condition $\sum_{v \in V(G)} f(v) \stackrel{2}{\equiv} 0$ when k is even. Let A, B be a partition of $V(G)$. If $e_G(A) + e_G(B) \geq k - 1$ when k is odd and $e_G(A) + e_G(B) \geq k/2 - 1$ when k is even, then G has a factor M such that*

$$\sum_{v \in A} (f(v) - d_M(v)) \stackrel{k}{\equiv} \sum_{v \in B} (f(v) - d_M(v)),$$

and for every edge xy belonging to $E(M)$, either $x, y \in A$ or $x, y \in B$.

Proof. Let M be a spanning subgraph of G consisting of some edges whose ends either lie in A or B . For each vertex v , define $f'(v) = f(v) - d_M(v)$. Obviously, $\sum_{v \in A} f'(v) - \sum_{v \in B} f'(v) \stackrel{k}{\equiv} \sum_{v \in A} f(v) - \sum_{v \in B} f(v) - 2n_a + 2(e_G(B) - n_b)$, where n_a is the number of edges of M whose ends lie in the set A , and $e_G(B) - n_b$ is the number of edges of M whose ends lie in the set B . Obviously, $0 \leq n_a \leq e_G(A)$ and $0 \leq n_b \leq e_G(B)$. First, suppose that k is odd. Since $e_G(A) + e_G(B) \geq k - 1$, the graph M can be chosen such that $2(n_a + n_b) \stackrel{k}{\equiv} \sum_{v \in A} f(v) - \sum_{v \in B} f(v) + 2e_G(B)$. Next, suppose that k is even. By the assumption, we can conclude that $\sum_{v \in A} f'(v) - \sum_{v \in B} f'(v) \stackrel{2}{\equiv} \sum_{v \in A} f(v) - \sum_{v \in B} f(v) + 2e_G(B) \stackrel{2}{\equiv} 0$. Therefore, we have $\frac{1}{2}(\sum_{v \in A} f'(v) - \sum_{v \in B} f'(v)) \stackrel{k/2}{\equiv} \frac{1}{2}(\sum_{v \in A} f(v) - \sum_{v \in B} f(v) + 2e_G(B)) - n_a - n_b$. Since $e_G(A) + e_G(B) \geq k/2 - 1$, similarly M can be chosen such that $n_a + n_b \stackrel{k/2}{\equiv} \frac{1}{2}(\sum_{v \in A} f(v) - \sum_{v \in B} f(v) + 2e_G(B))$. On the other hand, in both cases, M can be chosen such that f' is compatible by G . Hence the lemma holds. \square

Lemma 6. *Every graph G with $|E(G)| \geq 1$ can be decomposed into two factors G_1 and G_2 such that*

$|E(G_2)| \leq |E(G_1)| \leq |E(G_2)| + 1$ and for each $v \in V(G_i)$,

$$\lceil \frac{d_G(v)}{2} \rceil - 1 \leq d_{G_i}(v) \leq \lfloor \frac{d_G(v)}{2} \rfloor + 1.$$

Furthermore, for an edge $xy \in E(G_1)$, we have $d_{G_1}(x) \geq \lceil \frac{d_G(x)}{2} \rceil$ and $d_{G_1}(y) \geq \lceil \frac{d_G(y)}{2} \rceil$.

Proof. For $|V(G)| = 2$, the proof is straightforward. So, suppose $|V(G)| \geq 3$. Also, it suffices we prove the case that G is connected. If G has a cycle, then let xy be an edge of G which $G - xy$ is connected and define $G' = G - xy$; otherwise let xy be an edge of G with $d_G(x) = 1$ and define $G' = G - x$. Add a matching M to G' such that the resulting graph G'' whose degrees are even. Take an Eulerian tour for G'' with an initial vertex u distinct from x, y , walk on it and alternatively color the edges of G' red and blue by starting with the red color; if u is incident with an edge e of M , then start with the edge e . Finally, color the edge xy blue. Take G_1 to be the factor with all edges having the blue color and take G_2 to be the factor with all edges having the red color. It is easy to verify these graphs are the desired factors. \square

Lemma 7.([29]) *Every $(2m-1)$ -edge-connected graph has an m -edge-connected bipartite factor, where m is a positive integer.*

Proof of Theorem 28. By Lemma 7, the graph G has a $(3k-3)$ -edge-connected bipartite factor \mathcal{H} with maximum $|E(\mathcal{H})|$ and with the bipartition (A, B) . Set $\xi_k = k-1$ when k is odd and $\xi_k = k/2 - 1$ when k is even. Let \mathcal{G} be the factor of G with $E(\mathcal{G}) = E(G) \setminus E(\mathcal{H})$. Note that $|E(\mathcal{G})| \geq bi(G) \geq \xi_k$. If $|E(\mathcal{G})| = \xi_k$, then let M be a factor of \mathcal{G} and define F to be the factor of G with no edges. Otherwise, decompose \mathcal{G} into two factors \mathcal{G}_1 and \mathcal{G}_2 with the properties described in Lemma 6. Let W_1 and W_2 be two factors of G with $E(W_1) \subseteq E(\mathcal{G}_1)$ and $E(W_2) \subseteq E(\mathcal{G}_2)$ which $|W_1 \cup W_2| = \xi_k$ and $xy \in E(W_1)$. Since $|E(\mathcal{G})| > \xi_k$ and $|E(\mathcal{G}_2)| \leq |E(\mathcal{G}_1)| \leq |E(\mathcal{G}_2)| + 1$, we may assume that $|W_1| = \lceil \xi_k/2 \rceil$ and $|W_2| = \lfloor \xi_k/2 \rfloor$. In this case, define $F = \mathcal{G}_1 - E(W_1)$ and let M be a factor of \mathcal{G} with $E(M) \subseteq W_1 \cup W_2$. Now, for each vertex v , define $f'(v) = f(v) - d_F(v) - d_M(v)$. By applying Lemma 5 to $W_1 \cup W_2$, the graph M can be chosen such that f' is compatibly by \mathcal{H} . Let H' be an f' -factor of \mathcal{H} with the properties stated in Theorem 23. Set $H = \mathcal{H}' \cup F \cup M$. Obviously, H is an f -factor. Pick $v \in V(G)$. First assume that $|E(\mathcal{G})| = \xi_k$. Thus

$$d_H(v) \leq \lceil \frac{d_{\mathcal{H}}(v)}{2} \rceil + (k-1) + d_M(v) \leq \lceil \frac{d_G(v) - d_M(v)}{2} \rceil + (k-1) + d_M(v) \leq \lceil \frac{d_G(v)}{2} \rceil + k + \xi_k/2,$$

and

$$d_H(v) \geq \lfloor \frac{d_{\mathcal{H}}(v)}{2} \rfloor - (k-1) + d_M(v) \geq \lfloor \frac{d_G(v) - d_{\mathcal{G}}(v)}{2} \rfloor - (k-1) + d_M(v) \geq \lfloor \frac{d_G(v)}{2} \rfloor - k - \xi_k/2,$$

Next, assume that $|E(\mathcal{G})| > \xi_k$. Therefore,

$$d_H(v) \leq \lceil \frac{d_{\mathcal{H}}(v)}{2} \rceil + (k-1) + \lfloor \frac{d_{\mathcal{G}}(v)}{2} \rfloor + 1 + d_{W_2}(v) \leq \lceil \frac{d_G(v)}{2} \rceil + k + \xi_k/2,$$

and, if $d_{W_1}(v) \leq \xi_k/2$, then we have

$$d_H(v) \geq \lfloor \frac{d_{\mathcal{H}}(v)}{2} \rfloor - (k-1) + \lceil \frac{d_G(v)}{2} \rceil - 1 - d_{W_1}(v) \geq \lfloor \frac{d_G(v)}{2} \rfloor - k - \xi_k/2.$$

For the case $d_{W_1}(v) > \xi_k/2$, we can conclude that $v = x$ or $v = y$, because $xy \in E(W_1)$ and $|E(W_1)| = \lceil \xi_k/2 \rceil$. In this case, we also have $d_{G_1}(v) \geq \lceil \frac{d_G(v)}{2} \rceil$ which can complete the proof. \square

Note that another version of Theorem 28 could be provided depending on tree-connectivity condition. To establish it, we only require to replace the following lemma instead of Lemma 7. In addition, the factor H can be found with high tree-connectivity. To do that, we only require to replace Theorem 27 instead of Theorem 23. We leave the details for interested readers. Note that also the bound in Theorem 28 could be refined about $k/8$ and $k/4$ based on Theorem 8.

Lemma 8. *Every $2m$ -tree-connected graph G has an m -tree-connected bipartite factor.*

Proof. Let F be a bipartite spanning subgraph with maximum $|E(F)|$. First, we claim that for all vertex sets $X \subseteq V(F)$, $d_F(X) \geq d_G(X)/2$. This fact was implicitly utilized in the proof of Proposition 1 in [29]. To see this, suppose otherwise that $d_F(X) < d_G(X)/2$, for a vertex set X . Take E_X to be the set of all edges of G with exactly one end in X . Thus we have $|E_X \setminus E(F)| \geq d_G(X)/2$. It is not difficult to see that the spanning subgraph of G with the edge set $(E(F) \setminus E_X) \cup (E_X \setminus E(F))$ is also bipartite and has more edges than F which is a contradiction, as desired. Now, let X_1, \dots, X_t be a partition of the vertices of G . By Proposition 8 in Subsection 4.1, we have

$$\sum_{1 \leq i \leq t} d_F(X_i) \geq \frac{1}{2} \sum_{1 \leq i \leq t} d_G(X_i) \geq \frac{1}{2} (4m(t-1)) = 2m(t-1).$$

Again, by applying Proposition 8, the graph F has m edge-disjoint spanning trees. \square

4 Decomposing highly edge-connected graphs into two highly tree-connected factors

In this section, we turn our attention to investigate highly edge-connected factors by discounting modulo conditions on degrees. The results will be shown in this section generalize and refine the recent results in [1] and some of them were frequently used in the former section. However, the results in this section hold for graphs with loops, we leave it because it requires revising definition of $d_G(X)$ for every vertex set X consisting of a single vertex. In this section, we mainly deal with two types of spanning trees as follows: a directed spanning tree T with the vertex u is said to be **out-branching rooted at u** , if $d_T^-(u) = 0$ and $d_T^-(v) = 1$ for all vertices v with $v \neq u$; also T is said to be **in-branching rooted at u** , if $d_T^+(u) = 0$ and $d_T^+(v) = 1$ for all vertices v with $v \neq u$.

4.1 Preliminaries

The next proposition provides a simple but fruitful relationship between lifting operations and the existence of edge-disjoint branchings. We will apply it in the later subsections.

Proposition 7 *Let H be a graph obtained from G by lifting two incident edges. If H has an orientation and m edge-disjoint branchings, then G , in the orientation induced by H , has m edge-disjoint branchings with the same roots, by retaining the property of out-branching and in-branching.*

Proof. Let xu and uy be two edges of G , and assume that H is obtained from G by lifting xu and yu . Let T_1, \dots, T_m be m edge-disjoint branchings of H . If xu and yu are parallel, then T_1, \dots, T_m are the desired branchings of G . So, suppose xu and yu are not parallel. If $xy \notin E(T_1) \cup \dots \cup E(T_m)$, then T_1, \dots, T_m are again the desired branchings of G . So, suppose $xy \in E(T_1) \cup \dots \cup E(T_m)$. We may assume that $xy \in E(T_1)$ and the edge xy is directed from x to y . First, assume that T_1 is an out-branching. If u and x lie in the same component of $T_1 - xy$, then $\mathcal{T}_1 = T_1 - xy + uy$ is an out-branching of G with the same root of T_1 . Now, suppose that u and y lie in the same component of $T_1 - xy$. Then, there exists a unique directed path P of T_1 from y to u with the end edge zu . Thus, $\mathcal{T}_1 = T_1 - xy + uy - zu + xu$ is an out-branching of G with the same root of T_1 . Next, assume that T_1 is an in-branching. If u and y lie in the same component of $T_1 - xy$, then $\mathcal{T}_1 = T_1 - xy + xu$ is an in-branching of G with the same root of T_1 . Now, suppose that u and x lie in the same component of $T_1 - xy$. Then, there exists a unique directed path P of T_1 from u to x with the initial edge uz . Thus, $\mathcal{T}_1 = T_1 - xy + xu - uz + uy$ is an in-branching of G with the same root of T_1 . In the four former cases, $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_m$ are the desired branchings of G . \square

Proposition 8 ([25, 35]) *A graph G is m -tree-connected if and only if $\sum_{1 \leq i \leq t} d_G(X_i) \geq 2m(t-1)$, for every partition X_1, \dots, X_t of $V(G)$.*

4.2 Edge-disjoint branchings

In this subsection, we establish the following basic tool which will be used several times in the later subsections. Note that by restricting our attention to branchings with a prescribed root, the following theorem can easily be obtained from a combination of two main results in [11, 24]. The following theorem allows us to distribute the roots on arbitrary vertices.

Theorem 29. *Let G be a $2m$ -edge-connected graph with $z_0 \in V(G)$. For each vertex v , let $r(v)$ be a nonnegative integer. If $\sum_{v \in V(G)} r(v) = m$, then G has*

- (i) *an orientation and m edge-disjoint out-branchings such that each vertex v is the root of $r(v)$*

out-branchings, and also

$$d_G^+(v) \leq \begin{cases} \lfloor \frac{d_G(v)}{2} \rfloor & \text{if } v = z_0; \\ \lceil \frac{d_G(v)}{2} \rceil & \text{otherwise.} \end{cases}$$

(ii) an orientation and m edge-disjoint in-branchings such that each vertex v is the root of $r(v)$ in-branchings, and also

$$d_G^-(v) \leq \begin{cases} \lfloor \frac{d_G(v)}{2} \rfloor & \text{if } v = z_0; \\ \lceil \frac{d_G(v)}{2} \rceil & \text{otherwise.} \end{cases}$$

The first part of following result first found by Catlin [8] which refines the result of Nash-Williams [25] and Tutte [35]. By utilizing submodular inequality on vertex sets, we slightly improve it for applying in the proof Theorem 29.

Lemma 9. *Every $2m$ -edge-connected G with $|V(G)| \geq 2$ has an m -tree-connected factor H excluding the edges of a given arbitrary factor M of size m . In particular, if G has a vertex z_0 of odd degree, then H can be found excluding an edge e incident with z_0 that lies in $E(G) \setminus E(M)$. In addition, if $d_G(z_0) = 2m + 1$ and the edges of M are incident with z_0 , then e can be selected arbitrary.*

Proof. Let M be a factor of G of size m . Since G is $2m$ -edge-connected, for every vertex set X with $\emptyset \subsetneq X \subsetneq V(G)$, we have $d_G(X) \geq 2m$. Let z_0 be a vertex of odd degree. Suppose the theorem is false. Let $z_0 z_1 \in E(G) \setminus E(M)$ be an edge incident with z_0 . Since $G - E(M) - z_0 z_1$ is not m -tree-connected, by Proposition 8, there is a vertex set X_1 such that $z_0 \in X_1$, $z_1 \notin X_1$, $d_G(X_1) = 2m$, and any edge of M has at most one end in X_1 . If $d_G(z_0) = 2m + 1$ and all edges of M are incident with z_0 , then E forms an edge cut of size $2m - 1$, which is impossible; where E is the union of the set of edges incident with z_0 and the set of edges with exactly one end in X_1 apart from the edges belonging to $M \cup \{z_0 z_1\}$. Consider X_1 with minimum number of vertices. Since $d_G(z_0) > 2m$, there is an edge $z_0 z_2$ incident with z_0 distinct from $z_0 z_1$ that $z_2 \in X_1$. Corresponding to the edge $z_0 z_2$, again there is a vertex set X_2 such that $z_0 \in X_2$, $z_2 \notin X_2$, $d_G(X_2) = 2m$, and any edge of M has at most one end in X_2 . Thus

$$2m \leq \begin{cases} d_G(X_1 \cup X_2), & \text{if } X_1 \cup X_2 \neq V(G); \\ 2e_G(X_1 \setminus X_2, X_2 \setminus X_1), & \text{if } X_1 \cup X_2 = V(G). \end{cases}$$

These inequalities imply that

$$\begin{aligned} 4m &= d_G(X_1) + d_G(X_2) \\ &= d_G(X_1 \cap X_2) + d_G(X_1 \cup X_2) + 2e_G(X_1 \setminus X_2, X_2 \setminus X_1) \geq 4m, \end{aligned}$$

Thus we have $d_G(X_1 \cap X_2) = 2m = d_G(X_1 \cup X_2) + 2e_G(X_1 \setminus X_2, X_2 \setminus X_1)$. Since $\{z_0\} \subseteq X_1 \cap X_2 \subseteq X_1 \setminus \{z_2\} \subsetneq X_1$, we arrive at a contradiction, as desired. \square

Proof of Theorem 29. The proofs of two parts are similar and so we only prove the first one. For $m = 0$, the proof is straightforward. Suppose $m \geq 1$. We proceed by induction on the sum of all $d_G(v) - 2m - 1$ taken over all vertices v with $d_G(v) \geq 2m + 2$. First, assume that for each vertex v , $d_G(v) \leq 2m + 1$. Since G has minimum degree at least $2m$, there is a factor M of size m with an orientation such that for each vertex v , $d_M^-(v) = r(v)$. By Lemma 9, the graph $G - E(M)$ has m edge-disjoint spanning trees T_1, \dots, T_m . In particular, if the vertex z_0 has odd degree, then these trees can be found excluding an edge $e \in V(G) \setminus E(M)$ incident with z_0 . In this case, direct the edge e toward z_0 . Also, orient every T_i as an out-branching such that each vertex v is the root of $r(v)$ such branchings. Finally, orient all remaining edges of G arbitrary. With respect to this orientation, for each vertex v , we have $d_{T_1 \cup \dots \cup T_m}^-(v) = m - r(v)$ and so $d_G^-(v) \geq d_M^-(v) + d_{T_1 \cup \dots \cup T_m}^-(v) = r(v) + (m - r(v)) = m = \lfloor \frac{d_G(v)}{2} \rfloor$. This implies $d_G^+(v) \leq \lceil \frac{d_G(v)}{2} \rceil$. In particular, if the vertex z_0 has odd degree, then $d_G^-(z_0) \geq d_M^-(z_0) + d_{T_1 \cup \dots \cup T_m}^-(z_0) + 1 = r(z_0) + (m - r(z_0)) + 1 = m + 1 = \lceil \frac{d_G(z_0)}{2} \rceil$, which implies $d_G^+(z_0) \leq \lfloor \frac{d_G(z_0)}{2} \rfloor$. Now, assume that there is a vertex u with $d_G(u) \geq 2m + 2$. By Proposition 1, there are two edges xu and yu of G incident with u such that by lifting them the resulting graph H is still $2m$ -edge-connected. By the induction hypothesis, the graph H has an orientation and m edge-disjoint out-branchings such that each vertex v is the root of $r(v)$ out-branchings, $d_H^+(v) \leq \lceil \frac{d_H(v)}{2} \rceil$, and $d_H^+(z_0) \leq \lfloor \frac{d_H(z_0)}{2} \rfloor$. This orientation of H induces an orientation for G such that $d_G^+(u) = d_H^+(u) + 1$, $d_G^+(v) = d_H^+(v)$ for any $v \in V(G) \setminus \{u, x\}$, and

$$d_G^+(x) = \begin{cases} d_H^+(x) + 1, & \text{if } xu \text{ and } yu \text{ are parallel;} \\ d_H^+(x), & \text{if } xu \text{ and } yu \text{ are not parallel.} \end{cases}$$

For instance, for the vertex u , we have $d_G^+(u) \leq \lceil \frac{d_H(u)}{2} \rceil + 1 = \lceil \frac{d_G(u)}{2} \rceil$. With respect to Proposition 7, this orientation of G is the desired orientation. \square

Note that the Theorem 29 could be modified to a stronger version by selecting the factor M from G . To prove it, we only require to implement the lifting operation without selecting the edges of M ; see [13, Theorem B]. In addition, using such lifting operations, Lemma 9 could be modified to the following stronger version.

Theorem 30. *Every $2m$ -edge-connected G with $z_0 \in V(G)$ and with $|V(G)| \geq 2$ has an m -tree-connected factor H excluding the edges of a given arbitrary factor M of size m such that for each vertex v ,*

$$d_H(v) \leq \begin{cases} \lfloor \frac{d_G(z_0)}{2} \rfloor + m - d_M(z_0), & \text{if } v = z_0; \\ \lceil \frac{d_G(v)}{2} \rceil + m - d_M(v), & \text{otherwise.} \end{cases}$$

Proof. For $m = 0$, the proof is trivial. So, suppose $m \geq 1$. By induction on the sum of all $d_G(v) - 2m - 1$ taken over all vertices v with $d_G(v) \geq 2m + 2$. If for each vertex v , $d_G(v) \leq 2m + 1$, then the proof can be obtained directly from Lemma 9. Now, assume that there is a vertex u

with $d_G(u) \geq 2m + 2$. As we have remarked, there are two edges xu and yu of G distinct from the edges of M incident with u such that by lifting them the resulting graph G' is still $2m$ -edge-connected. By the induction hypothesis, G' has a factor H' consisting of m edge-disjoint spanning trees T_1, \dots, T_m excluding the edges of M such that for each vertex v , $d_{H'}(v) \leq \lceil \frac{d_{G'}(v)}{2} \rceil + m - d_M(v)$, and $d_{H'}(z_0) \leq \lfloor \frac{d_{G'}(z_0)}{2} \rfloor + m - d_M(z_0)$. If xu and yu are parallel or $xy \notin E(H')$, the theorem can easily be proved. Thus we may assume that $xy \in E(T_1)$. If x and u lie in the same component of $T_1 - xy$, define $\mathcal{T}_1 = T_1 - xy + uy$; otherwise, define $\mathcal{T}_1 = T_1 - xy + xu$. It is easy to see that the factor H consisting of m edge-disjoint spanning trees $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \dots \cup \mathcal{T}_m$ satisfies the theorem. \square

4.3 Highly edge-connected factors with bounded degrees

In this subsection, we will prove Propositions 5 and 6, as we have promised. A major step toward this goal is provided by the following theorem.

Theorem 31. *Let G be a directed graph with two edge-disjoint factors F_1 and F_2 . For each vertex v , take $s_1(v)$ and $s_2(v)$ to be two nonnegative integers with $s_1(v) + s_2(v) \geq d_G^+(v) - d_G^-(v)$. If G is connected and $|E(F_1 \cup F_2)| \geq 1$, then G can be decomposed into two factors G_1 and G_2 such that each G_i contains F_i and for each vertex v ,*

$$\begin{aligned} d_G^+(v) - d_{F_2}^+(v) - s_2(v) &\leq d_{G_1}(v) \leq d_G^-(v) + d_{F_1}^+(v) + s_1(v), \\ d_G^+(v) - d_{F_1}^+(v) - s_1(v) &\leq d_{G_2}(v) \leq d_G^-(v) + d_{F_2}^+(v) + s_2(v). \end{aligned}$$

Proof. Add some new directed edges to G such that for each vertex v of the resulting graph \mathcal{G} , we have $d_{\mathcal{G}}^+(v) = d_G^-(v)$, and also $d_M(v) = |d_G^+(v) - d_G^-(v)|$, where M is the factor of \mathcal{G} consisting of all such new edges. We may assume that $|E(F_1)| \geq 1$. Since \mathcal{G} is connected, it has a directed Eulerian tour e_1, \dots, e_t by starting at an edge e_1 with $e_1 \in E(F_1)$. For a vertex v with $d_G^+(v) > d_G^-(v)$, let $\omega_j(v)$ be the j -th incoming edge of v in the tour with $\omega_j(v) = e_{i-1} \in E(M)$ and $e_i \notin E(F_1 \cup F_2)$, where $e_{t+1} = e_1$ and $j \geq 1$. Set W_v to be the set of all such edges. Let \mathcal{W}_v be the set of all edges e_i incident with v such that $e_{i-1} \in W_v$, where $i \geq 2$. In plus, for all other vertices v with $d_G^+(v) \leq d_G^-(v)$, take W_v and \mathcal{W}_v to be the empty set. Note that for each vertex v , we have $|\mathcal{W}_v| = |W_v| \leq \max\{0, d_G^+(v) - d_G^-(v)\} \leq s_1(v) + s_2(v)$. Now, we shall construct the nested spanning subgraphs H_0, \dots, H_t of G . Let H_0 be the graph with no edges. Take i to be an integer with $1 \leq i \leq t$. Now, with respect to the following rules inductively construct H_i from H_{i-1} ,

- ▷ If $e_i \notin E(M) \cup E(F_1 \cup F_2)$,
- ▷ If $e_{i-1} \in W_v$ with $e_{i-1} = \omega_j(v)$,
 - ▷ If $j \leq s_2(v)$, set $H_i = H_{i-1}$.
 - ▷ Else if $j > s_2(v)$, set $H_i = H_{i-1} + e_i$.

- ▷ Else
 - ▷ If $e_{i-1} \in E(H_{i-1})$, set $H_i = H_{i-1}$.
 - ▷ Else if $e_{i-1} \notin E(H_{i-1})$, set $H_i = H_{i-1} + e_i$.
- ▷ Else if $e_i \in E(F_1)$, set $H_i = H_{i-1} + e_i$.
- ▷ Else if $e_i \in E(M) \cup E(F_2)$, set $H_i = H_{i-1}$.

Put $H = H_t$. Obviously, H is a spanning subgraph of G including F_1 excluding F_2 . Pick $v \in V(G)$. Let $E_H(v)$ be the set of all edges of H incident with v , and let $Q(v)$ be the set of pair of edges of G incident with v as e_j, e_{j+1} such that $e_j \notin W_v$ and $e_{j+1} \notin E(F_1 \cup F_2) \cup E(M)$. Thus, we have

$$|Q(v)|/2 = d_G^+(v) - d_{F_1}^+(v) - d_{F_2}^+(v) - |W_v|.$$

According to the construction of H , it is not difficult to verify that $|E_H(v) \cap Q(v)| = |Q(v)|/2$, $|E_H(v) \cap E(F_2)| = 0$, and $|E_H(v) \cap E(F_1) \setminus Q(v)| \geq d_{F_1}^+(v)$, $|E_H(v) \cap W_v| \geq |W_v| - s_2$. Therefore, one can conclude that

$$d_H(v) \geq |Q(v)|/2 + d_{F_1}^+(v) + |E_H(v) \cap W_v| \geq d_G^+(v) - d_{F_2}^+(v) - s_2(v).$$

Also, we have

$$d_H(v) \leq d_G(v) - |Q(v)|/2 - d_{F_2}^+(v) - \min\{s_2, |W_v|\} \leq d_G^-(v) + d_{F_1}^+(v) + s_1(v).$$

Now, by setting $G_1 = H$ and $G_2 = G - E(G_1)$, the proof can be completed. \square

A slightly stronger but more complicated version of Theorem 31 is give in the following corollary.

Corollary 17. *Let G be a directed graph with two edge-disjoint factors F_1 and F_2 . Let \mathcal{F}_i be a factor of F_i . For each vertex v , take $s_1(v)$ and $s_2(v)$ to be two nonnegative integers with $s_1(v) + s_2(v) \geq (d_G^+(v) - d_{\mathcal{F}_1}^+(v) - d_{\mathcal{F}_2}^+(v)) - (d_G^-(v) - d_{\mathcal{F}_1}^-(v) - d_{\mathcal{F}_2}^-(v))$. If $G - E(\mathcal{F}_1 \cup \mathcal{F}_2)$ is connected and $|E(F_1 \cup F_2)| > |E(\mathcal{F}_1 \cup \mathcal{F}_2)|$, then G can be decomposed into two factors G_1 and G_2 such that each G_i contains F_i and for each vertex v ,*

$$\begin{aligned} d_G^+(v) - d_{F_2}^+(v) - s_2(v) + d_{\mathcal{F}_1}^-(v) &\leq d_{G_1}(v) \leq d_G^-(v) + d_{F_1}^+(v) + s_1(v) - d_{\mathcal{F}_2}^-(v), \\ d_G^+(v) - d_{F_1}^+(v) - s_1(v) + d_{\mathcal{F}_2}^-(v) &\leq d_{G_2}(v) \leq d_G^-(v) + d_{F_2}^+(v) + s_2(v) - d_{\mathcal{F}_1}^-(v). \end{aligned}$$

Proof. Apply Theorem 31 to the graph $G - E(\mathcal{F}_1 \cup \mathcal{F}_2)$ along with factors $F_1 - E(\mathcal{F}_1)$ and $F_2 - E(\mathcal{F}_2)$. \square

Corollary 18. *Let G be a directed graph with two edge-disjoint factors F_1 and F_2 . If G is connected and $|E(F_1 \cup F_2)| \geq 1$, then G can be decomposed into two factors G_1 and G_2 such that each G_i contains*

F_i and for each vertex v with $j \in \{1, 2\}$ that $j \neq i$,

$$\begin{aligned} \lfloor \frac{d_G(v)}{2} \rfloor - d_{F_j}^+(v) &\leq d_{G_i}(v) \leq \lceil \frac{d_G(v)}{2} \rceil + d_{F_i}^+(v), & \text{if } d_G^+(v) > d_G^-(v); \\ d_G^+(v) - d_{F_j}^+(v) &\leq d_{G_i}(v) \leq d_G^-(v) + d_{F_i}^+(v), & \text{if } d_G^+(v) \leq d_G^-(v). \end{aligned}$$

Proof. For each vertex v , set $s_2(v) = \lceil (d_G^+(v) - d_G^-(v))/2 \rceil$ and $s_1(v) = \lfloor (d_G^+(v) - d_G^-(v))/2 \rfloor$, if $d_G^+(v) > d_G^-(v)$. Otherwise, set $s_1(v) = s_2(v) = 0$. Now, it is enough to apply Theorem 31. \square

Corollary 19. Every $(2m_1 + 2m_2)$ -edge-connected graph G with $m_1 + m_2 \geq 1$ can be decomposed into two factors G_1 and G_2 such that each G_i is m_i -tree-connected and for each vertex v ,

$$\begin{aligned} \lfloor \frac{d_G(v)}{2} \rfloor - m_2 + r_2(v) &\leq d_{G_1}(v) \leq \lceil \frac{d_G(v)}{2} \rceil + m_1 - r_1(v), \\ \lfloor \frac{d_G(v)}{2} \rfloor - m_1 + r_1(v) &\leq d_{G_2}(v) \leq \lceil \frac{d_G(v)}{2} \rceil + m_2 - r_2(v). \end{aligned}$$

where $r_1(v)$ and $r_2(v)$ are arbitrary nonnegative integers with $\sum_{v \in V(G)} r_i(v) = m_i$.

Proof. By Theorem 29, the graph G has an orientation and two edge-disjoint factors F_1 and F_2 such that each F_i consists of m_i edge-disjoint in-branchings, and for each vertex v , $d_{F_i}^+(v) = m_i - r_i(v)$, $d_G^-(v) \leq \lceil \frac{d_G(v)}{2} \rceil$. Now, it is enough to apply Corollary 18. \square

Corollary 20. Let G be a directed graph with the factor F . If G is connected and $|E(F)| \geq 1$, then G has a factor H containing F such that for each vertex v ,

$$\begin{aligned} \lfloor \frac{d_G(v)}{2} \rfloor &\leq d_H(v) \leq \lfloor \frac{d_G(v)}{2} \rfloor + d_F^+(v) & \text{if } d_G^+(v) > d_G^-(v); \\ d_G^+(v) &\leq d_H(v) \leq d_G^-(v) + d_F^+(v), & \text{if } d_G^+(v) \leq d_G^-(v) \end{aligned}$$

Proof. For each vertex v , set $s_2(v) = \lceil (d_G^+(v) - d_G^-(v))/2 \rceil$ and $s_1(v) = \lfloor (d_G^+(v) - d_G^-(v))/2 \rfloor$, if $d_G^+(v) > d_G^-(v)$. Otherwise, set $s_1(v) = s_2(v) = 0$. Now, it is enough to apply Theorem 31. \square

A supplement for Theorem 8 in [1] is given as the following result.

Corollary 21. Every $2m$ -edge-connected graph G with $z_0 \in V(G)$ and $m > 0$ has an m -tree-connected factor H such that for each vertex v ,

$$\lfloor \frac{d_G(v)}{2} \rfloor \leq d_H(v) \leq \begin{cases} \lfloor \frac{d_G(v)}{2} \rfloor + m - r(v), & \text{if } v = z_0; \\ \lceil \frac{d_G(v)}{2} \rceil + m - r(v), & \text{otherwise.} \end{cases}$$

where $r(v)$ is an arbitrary nonnegative integer with $\sum_{v \in V(G)} r(v) = m$.

Proof. By Theorem 29, the graph G has an orientation and m edge-disjoint in-branchings such that each vertex v is the root of $r(v)$ in-branchings, $d_G^-(v) \leq \lceil \frac{d_G(v)}{2} \rceil$, and $d_G^-(z_0) \leq \lfloor \frac{d_G(z_0)}{2} \rfloor$. Let F be the union of these in-branchings. Note that for each vertex v , $d_F^+(v) = m - r(v)$. Now, Corollary 20 implies that F can be extended to a factor H with the desired properties. \square

Now, we are ready to prove Proposition 5 by a stronger version.

Theorem 32. *Let G be $(2m_1 + 2m_2)$ -edge-connected graph. For each vertex v , take $r_1(v)$, $r_2(v)$, and $s(v)$ to be nonnegative integers with $\sum_{v \in V(G)} r_1(v) = m_1$, $\sum_{v \in V(G)} r_2(v) = m_2$, and $s(v) \leq m_1 + r_2(v)$. If $m_2 \geq 1$, then G has two edge-disjoint factors M_1 and M_2 such that M_1 is m_1 -tree-connected, M_2 can be transformed into an m_2 -tree-connected graph L by alternatively lifting operations, and for each vertex v ,*

$$\begin{aligned} (i) \quad & d_{M_1}(v) + \frac{d_{M_2}(v) - d_L(v)}{2} \geq \lfloor \frac{d_G(v)}{2} \rfloor - m_1 - m_2 + s(v), \\ (ii) \quad & d_{M_1}(v) + \frac{d_{M_2}(v) + d_L(v)}{2} \leq \lceil \frac{d_G(v)}{2} \rceil + m_1 + m_2 + s(v) - r_1(v) - r_2(v). \end{aligned}$$

Furthermore, for an arbitrary vertex z_0 we can have $d_{M_1}(z_0) + \frac{d_{M_2}(z_0) + d_L(z_0)}{2} \leq \lfloor \frac{d_G(z_0)}{2} \rfloor + m_1 + m_2 + s(z_0) - r_1(z_0) - r_2(z_0)$, and also $s(z_0) \leq m_1 + r_2(z_0) + 1$ when $d_G(z_0)$ is odd.

Proof. Let H be a $(2m_1 + 2m_2)$ -edge-connected graph obtained from G by alternatively lifting operations such that $\Delta(H) \leq 2m_1 + 2m_2 + 1$, using Proposition 1. First, assume that $m_1 \geq 1$. The graph H has an orientation and two edge-disjoint factors F and L such that F consists of m_1 edge-disjoint in-branchings, L consists of m_2 edge-disjoint in-branchings, and for each vertex v , $d_F^+(v) = m_1 - r_1(v)$, $d_L^+(v) = m_2 - r_2(v)$, $d_H^+(v) \geq m_1 + m_2$, and $d_H^+(z_0) \geq \lceil d_H(z_0)/2 \rceil$, using Theorem 29. Consider the orientation of G induced by the orientation of H . Let R be the factor of G obtained by reversing the lifting operations of Q where $Q = H - E(L)$. Note that for each vertex v , $d_Q^+(v) \geq \varphi(v) \geq s(v)$, where

$$\varphi(v) = \begin{cases} m_1 + r_2(v) + 1, & \text{if } v = z_0 \text{ and has odd degree;} \\ m_1 + r_2(v), & \text{otherwise.} \end{cases}$$

By Proposition 7, the graph R has an m_1 -tree-connected factor \mathcal{F} such that for each vertex v , $d_{\mathcal{F}}^+(v) = d_F^+(v)$. Let M_2 be the factor of G obtained by reversing the lifting operations of L . Take \mathcal{R} to be the factor of G with $E(\mathcal{R}) = E(G) \setminus E(M_2)$ containing R . By Theorem 31, the graph \mathcal{R} has a factor M_1 containing \mathcal{F} such that for each vertex v ,

$$d_{\mathcal{R}}^+(v) - s_2(v) \leq d_{M_1}(v) \leq d_{\mathcal{R}}^-(v) + d_{\mathcal{F}}^+(v) + s_1(v),$$

where $s_1(v)$ and $s_2(v)$ are two nonnegative integers with

$$\begin{aligned} s_1(v) &= (d_Q^+(v) - \varphi(v)) + s(v), \\ s_2(v) &= \max\{0, d_Q^+(v) - d_Q^-(v) - s_1(v)\}, \\ s_1(v) + s_2(v) &\geq d_Q^+(v) - d_Q^-(v) = d_R^+(v) - d_R^-(v) = d_{\mathcal{R}}^+(v) - d_{\mathcal{R}}^-(v). \end{aligned}$$

Note that the resulting graphs have the following features:

$$E(F) \subseteq E(Q) \hookrightarrow E(R) \subseteq E(\mathcal{R}) \supseteq E(M_1) \supseteq E(\mathcal{F}) \quad \text{and} \quad E(L) \hookrightarrow E(M_2).$$

Now, for each vertex v , we have

$$\begin{aligned} d_{M_1}(v) + \frac{d_{M_2}(v) - d_L(v)}{2} &\geq d_{\mathcal{R}}^+(v) - s_2(v) + \frac{d_{M_2}(v) - d_L(v)}{2} \\ &\geq \frac{d_{\mathcal{R}}(v) - d_Q(v)}{2} + d_Q^+(v) - s_2(v) + \frac{d_{M_2}(v) - d_L(v)}{2} \\ &\geq \frac{d_G(v) - d_Q(v) - d_L(v)}{2} + d_Q^+(v) - s_2(v) \\ &\geq \lfloor \frac{d_G(v)}{2} \rfloor - m_1 - m_2 + \min\{d_Q^-(v) + s_1(v), d_Q^+(v)\} \\ &\geq \lfloor \frac{d_G(v)}{2} \rfloor - m_1 - m_2 + \min\{d_Q^-(v) + s(v), \varphi(v)\} \\ &\geq \lfloor \frac{d_G(v)}{2} \rfloor - m_1 - m_2 + s(v). \end{aligned}$$

Also,

$$\begin{aligned} d_{M_1}(v) + \frac{d_{M_2}(v) + d_L(v)}{2} &\leq d_{\mathcal{R}}^-(v) + d_{\mathcal{F}}^+(v) + s_1(v) + \frac{d_{M_2}(v) + d_L(v)}{2} \\ &\leq \frac{d_{\mathcal{R}}(v) - d_Q(v)}{2} + d_Q^-(v) + d_{\mathcal{F}}^+(v) + s_1(v) + \frac{d_{M_2}(v) + d_L(v)}{2} \\ &\leq \frac{d_{\mathcal{R}}(v) + d_Q(v)}{2} - d_Q^+(v) + d_{\mathcal{F}}^+(v) + s_1(v) + \frac{d_{M_2}(v) + d_L(v)}{2} \\ &\leq \frac{d_G(v) + d_Q(v) + d_L(v)}{2} + d_{\mathcal{F}}^+(v) + s_1(v) - d_Q^+(v) \\ &\leq \lceil \frac{d_G(v)}{2} \rceil + m_1 + m_2 + (m_1 - r_1(v)) + s(v) - \varphi(v) \\ &\leq \lceil \frac{d_G(v)}{2} \rceil + m_1 + m_2 - r_1(v) - r_2(v) + s(v). \end{aligned}$$

Now, assume that $m_1 = 0$. Since H has minimum degree at least $2m_2$, there is a factor Q with an orientation such that for each vertex v ,

$$d_Q^+(v) = \begin{cases} r_2(v) + 1, & \text{if } v = z_0 \text{ and has odd degree;} \\ r_2(v) & \text{otherwise,} \end{cases}$$

and the graph $H - E(Q)$ is m -tree-connected, using Lemma 9. Set $L = H - E(Q)$. Let R be the factor of G obtained by reversing the lifting operations of Q . For every edge xy in Q , consider the

trail P_{xy} in R with the end vertices x and y corresponding to xy . Now, for every edge xy of Q which is directed from x to y , alternatively color the edges of P_{xy} blue and red by starting at x such that precisely $s(v)$ trails are started with the blue color. Let M_1 be the factor of R consisting of all edges having the blue color. For each vertex v , we have

$$\frac{d_R(v) - d_Q(v)}{2} + s(v) \leq d_{M_1}(v) \leq \frac{d_R(v) + d_Q(v)}{2} - (d_Q^+(v) - s(v)).$$

Take $M_2 = G - E(R)$. Note that M_2 can be transformed into L by alternatively lifting operations. Thus

$$\begin{aligned} d_{M_1}(v) + \frac{d_{M_2}(v) - d_L(v)}{2} &\geq \frac{d_R(v) - d_Q(v)}{2} + s(v) + \frac{d_{M_2}(v) - d_L(v)}{2} \\ &\geq \frac{d_G(v) - d_Q(v) - d_L(v)}{2} + s(v) \\ &\geq \left\lfloor \frac{d_G(v)}{2} \right\rfloor - m_2 + s(v). \end{aligned}$$

Also,

$$\begin{aligned} d_{M_1}(v) + \frac{d_{M_2}(v) + d_L(v)}{2} &\leq \frac{d_R(v) + d_Q(v)}{2} - d_Q^+(v) + s(v) + \frac{d_{M_2}(v) + d_L(v)}{2} \\ &\leq \frac{d_G(v) + d_Q(v) + d_L(v)}{2} - d_Q^+(v) + s(v) \\ &\leq \left\lceil \frac{d_G(v)}{2} \right\rceil + m_2 - r_2(v) + s(v). \end{aligned}$$

Note that the extra conditions on z_0 can be derived similarly. These inequalities complete the proof. \square

As we have observed in the proof of Theorem 35, the result is rely on decomposition of a $(2m_1 + 2m_2)$ -edge-connected graph with maximum degree at most $2m_1 + 2m_2 + 1$. Therefore, to improve the lower bounds of Theorem 35 and consequently several results in Section 3, we pose the following problem.

Problem 1 *For any two positive numbers m_1 and m_2 find the smallest number $\phi(m_1, m_2)$ which the following holds: Every $(2m_1 + 2m_2)$ -edge-connected graph G with $\Delta(G) \leq 2m_1 + 2m_2 + 1$ can be decomposed into two factors G_1 and G_2 such that each G_i is m_i -tree-connected and $\Delta(G_2) \leq \phi(m_1, m_2)$.*

For the special case $m_1 = m_2 = 1$, we propose the following conjecture which can increase the lower bound of Theorem 19 with one, for odd vertices.

Conjecture 4. *Every 4-edge-connected graph G with $\Delta(G) \leq 5$ can be decomposed into two connected factors G_1 and G_2 such that $\Delta(G_2) \leq 3$.*

In the following, we want to prove Proposition 6 by a stronger version.

Theorem 33. *Every $(2m_1 + 2m_2)$ -edge-connected graph G with $m_1 \geq 1$ has two edge-disjoint factors G_1 and G_2 such that G_1 is m_1 -tree-connected, G_2 consists of m_2 edge-disjoint spanning trees, and for each vertex v at least one of the following conditions holds,*

- (i) $\lfloor \frac{d_G(v) - d_{G_2}(v)}{2} \rfloor \leq d_{G_1}(v) \leq \lfloor \frac{d_G(v) - d_{G_2}(v)}{2} \rfloor + m_1 - r_1(v),$
- (ii) $\lfloor \frac{d_G(v)}{2} \rfloor - m_2 \leq d_{G_1}(v) \leq d_{G_1}(v) + d_{G_2}(v) \leq \lceil \frac{d_G(v)}{2} \rceil + m_1 + m_2 - r_1(v) - r_2(v),$

where $r_1(v)$ and $r_2(v)$ are arbitrary nonnegative integers with $\sum_{v \in V(G)} r_i(v) = m_i$.

Proof. By Theorem 29, the graph G has an orientation and $m_1 + m_2$ edge-disjoint in-branchings such that each vertex v is the root of $r_1(v) + r_2(v)$ in-branchings and $d_G^-(v) \leq \lceil \frac{d_G(v)}{2} \rceil$. Let F be the union of m_1 edge-disjoint in-branchings where each vertex v is the root of $r_1(v)$ in-branchings, and let G_2 be the union of other m_2 edge-disjoint in-branchings. Set $\mathcal{G} = G - E(G_2)$. Note that for each vertex v , $d_F^+(v) = m_1 - r_1(v)$ and $d_{G_2}^+(v) = m_2 - r_2(v)$ and also $d_{\mathcal{G}}^+(v) = d_G^+(v) - d_{G_2}^+(v) \geq \lfloor \frac{d_G(v)}{2} \rfloor - m_2 + r_2(v)$. By Corollary 20, the graph \mathcal{G} has a factor G_1 containing F such that for each vertex v ,

$$\begin{cases} \lfloor \frac{d_{\mathcal{G}}(v)}{2} \rfloor \leq d_{G_1}(v) \leq \lfloor \frac{d_{\mathcal{G}}(v)}{2} \rfloor + d_F^+(v), & \text{if } d_{\mathcal{G}}^+(v) > d_G^-(v); \\ d_{\mathcal{G}}^+(v) \leq d_{G_1}(v) \leq d_G^-(v) + d_F^+(v), & \text{if } d_{\mathcal{G}}^+(v) \leq d_G^-(v). \end{cases}$$

Since $d_G^-(v) + d_{G_2}(v) = d_G^-(v) + d_{G_2}^+(v) \leq \lceil \frac{d_G(v)}{2} \rceil + m_2 - r_2(v)$, the proof can be completed. \square

Note that Theorem 23 could be rephrased as the following decomposition version which can be proved similarly to Proposition 2 in [31].

Theorem 34. *Let G be a bipartite graph with the bipartition (V_1, V_2) , let k be an integer, $k \geq 3$, and let $p : V(G) \rightarrow Z_k$ be a mapping with $|E(G)| \stackrel{k}{\equiv} \sum_{v \in V(G)} p(v)$. If G is $(3k - 3)$ -edge-connected, then it can be decomposed into two factors G_1 and G_2 such that for each $v \in V_i$, $d_{G_i}(v) \stackrel{k}{\equiv} p(v)$ and*

$$\lfloor \frac{d_G(v)}{2} \rfloor - (k - 1) \leq d_{G_i}(v) \leq \lceil \frac{d_G(v)}{2} \rceil + (k - 1).$$

This version may motivate one to investigate G_1 and G_2 with high edge-connectivity. Fortunately, this goal can be achieved by replacing a more complicated version of Propositions 5 and 6 depending on three parameters m_1 , m_2 , and m_3 as the following results. For proving them, we require the affection of both F_1 and F_2 of Theorem 31. We leave the details for interested readers.

Theorem 35. *Let G be $(2m_1 + 2m_2 + 2m_3)$ -edge-connected graph. For each vertex v , take $s(v)$ to be a nonnegative integer with $s(v) \leq m_1 + m_2$. If $m_3 \geq 1$, then G can be decomposed into edge-disjoint factors M_1 , M_2 and M_3 such that M_1 is m_1 -tree-connected, M_2 is m_2 -tree-connected, M_3 can be transformed into an m_3 -tree-connected graph L by alternatively lifting operations, and for each vertex v ,*

$$\begin{aligned}
(i) \quad & d_{M_1}(v) + \frac{d_{M_3}(v) - d_L(v)}{2} \geq \lfloor \frac{d_G(v)}{2} \rfloor - m_1 - 2m_2 - m_3 + s(v), \\
& d_{M_2}(v) + \frac{d_{M_3}(v) - d_L(v)}{2} \geq \lfloor \frac{d_G(v)}{2} \rfloor - m_1 - m_3 - s(v), \\
(ii) \quad & d_{M_1}(v) + \frac{d_{M_3}(v) + d_L(v)}{2} \leq \lceil \frac{d_G(v)}{2} \rceil + m_1 + m_3 + s(v), \\
& d_{M_2}(v) + \frac{d_{M_3}(v) + d_L(v)}{2} \leq \lceil \frac{d_G(v)}{2} \rceil + m_1 + 2m_2 + m_3 - s(v).
\end{aligned}$$

Theorem 36. *Every $(2m_1 + 2m_2 + 2m_3)$ -edge-connected graph G with $m_1 + m_2 \geq 1$ can be decomposed into edge-disjoint factors G_1, G_2 and G_3 such that G_1 is m_1 -tree-connected, G_2 is m_2 -tree-connected, and G_3 consists of m_3 edge-disjoint spanning trees, and for each vertex v at least one of the following conditions (i) and (ii) holds,*

$$\begin{aligned}
(i) \quad & \lfloor \frac{d_G(v) - d_{G_3}(v)}{2} \rfloor - m_2 \leq d_{G_1}(v) \leq \lceil \frac{d_G(v) - d_{G_3}(v)}{2} \rceil + m_1, \\
& \lfloor \frac{d_G(v) - d_{G_3}(v)}{2} \rfloor - m_1 \leq d_{G_2}(v) \leq \lceil \frac{d_G(v) - d_{G_3}(v)}{2} \rceil + m_2, \\
(ii) \quad & \lfloor \frac{d_G(v)}{2} \rfloor - m_2 - m_3 \leq d_{G_1}(v) \leq d_{G_1}(v) + d_{G_3}(v) \leq \lceil \frac{d_G(v)}{2} \rceil + m_1 + m_3, \\
& \lfloor \frac{d_G(v)}{2} \rfloor - m_1 - m_3 \leq d_{G_2}(v) \leq d_{G_2}(v) + d_{G_3}(v) \leq \lceil \frac{d_G(v)}{2} \rceil + m_2 + m_3.
\end{aligned}$$

4.4 Highly edge-connected factors using given lists on degrees

In this subsection, similarly to Theorem 31, we formulate the following theorem for decomposing graphs into two highly edge-connected factors which degrees of both factors are restricted by given lists. We here adopt the notation $L(v)$ for list of integers.

Theorem 37. *Let G be a directed graph with two edge-disjoint factors F_1 and F_2 , and let V_1 and V_2 be two disjoint vertex sets with $V_1 \cup V_2 = V(G)$. For each vertex v , take $s_1(v)$ and $s_2(v)$ to be two nonnegative integers and let $L(v) \subseteq \{s_1(v), \dots, d_G(v) - s_2(v)\}$, where*

$$\begin{cases} s_1(v) \leq d_{F_1}(v) \text{ and } s_2(v) \leq d_{F_2}(v), & \text{if } v \in V_1; \\ s_1(v) \leq d_{F_2}(v) \text{ and } s_2(v) \leq d_{F_1}(v), & \text{if } v \in V_2. \end{cases}$$

If for each vertex v ,

$$|L(v)| \geq d_G^+(v) + 1 + d_{F_1}^-(v) + d_{F_2}^-(v) - s_1(v) - s_2(v).$$

then G can be decomposed into two factors G_1 and G_2 such that each graph G_i contains F_i and for each $v \in V_i$, $d_{G_i}(v) \in L(v)$.

Proof. Put $F = F_1 \cup F_2$ and $H = G - E(F)$. For each vertex $v \in V_1$, define

$$L'(v) = \{l - d_{F_1}(v) : l \in L(v) \text{ and } d_{F_1}(v) \leq l \leq d_G(v) - d_{F_2}(v)\} \subseteq \{0, \dots, d_H(v)\}.$$

Thus we have

$$\begin{aligned} |L'(v)| &\geq |L(v)| - (d_{F_1}(v) - s_1(v)) - (d_{F_2}(v) - s_2(v)) \\ &\geq d_G^+(v) + 1 - d_{F_2}^+(v) - d_{F_1}^+(v) = d_H^+(v) + 1. \end{aligned}$$

For each vertex $v \in V_2$, define

$$L'(v) = \{d_G(v) - l - d_{F_1}(v) : l \in L(v) \text{ and } d_{F_2}(v) \leq l \leq d_G(v) - d_{F_1}(v)\} \subseteq \{0, \dots, d_H(v)\}.$$

Again, we have

$$\begin{aligned} |L'(v)| &\geq |L(v)| - (d_{F_2}(v) - s_1(v)) - (d_{F_1}(v) - s_2(v)) \\ &\geq d_G^+(v) + 1 - d_{F_1}^+(v) - d_{F_2}^+(v) = d_H^+(v) + 1. \end{aligned}$$

Therefore, by Theorem 2 in [14] (which was implicitly proved in [27]), the graph H has a factor H' such that for each vertex v , $d_{H'}(v) \in L'(v)$. Put $G_1 = H' \cup F_1$ and $G_2 = G - E(G_1)$ so that $E(G_2) \supseteq E(F_2)$. For each $v \in V_1$, we have $d_{G_1}(v) = d_{H'}(v) + d_{F_1}(v) \in L(v)$, and for each $v \in V_2$, we also have $d_{G_2}(v) = d_G(v) - (d_{H'}(v) + d_{F_1}(v)) \in L(v)$. Therefore, the graphs G_1 and G_2 are the desired factors. \square

A supplement for Theorem 4 in [1] is given as the following result.

Corollary 22. *Let G be a $2m$ -edge-connected graph with $z_0 \in V(G)$. For each vertex v , let $L(v) \subseteq \{m, \dots, d_G(v)\}$ and let $r(v)$ be a nonnegative integer with $\sum_{v \in V(G)} r(v) = m$. If for each vertex v ,*

$$|L(v)| \geq \begin{cases} \lfloor \frac{d_G(v)}{2} \rfloor + 1 - r(v), & \text{if } v = z_0; \\ \lceil \frac{d_G(v)}{2} \rceil + 1 - r(v), & \text{otherwise,} \end{cases}$$

then G contains an m -tree-connected factor H such that for each vertex v , $d_H(v) \in L(v)$.

Proof. By Theorem 29, the graph G has an orientation and m edge-disjoint out-branchings such that each vertex v is the root of $r(v)$ out-branchings, $d_G^+(v) \leq \lceil \frac{d_G(v)}{2} \rceil$, and $d_G^+(z_0) \leq \lfloor \frac{d_G(z_0)}{2} \rfloor$. Let F be the union of these out-branchings. Note that for each vertex v , $d_F^-(v) = m - r(v)$. Now, Theorem 37 implies that F can be extended to a factor H satisfying the theorem. \square

The result of Shirazi and Verstraëte [27] could be rephrased as the following version for $m = 0$.

Corollary 23. *Let G be a $4m$ -edge-connected graph and for each vertex v , let $L(v) \subseteq \{m, \dots, d_G(v) - m\}$. Take V_1 and V_2 to be disjoint vertex sets. If for each vertex v ,*

$$|L(v)| \geq \lceil \frac{d_G(v)}{2} \rceil + 1,$$

then G can be decomposed into two m -tree-connected factors G_1 and G_2 such that for each $v \in V_i$, $d_{G_i}(v) \in L(v)$.

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